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Evaluation of evidences for dynamic systems based on Bayes factors with an application

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Abstract. This paper deals with the computation of Bayes factors (BFs) based on sequential order statistics arising from homogeneous exponential populations. Explicit expressions for the BFs are derived from the chi-square and the Poisson distribution functions. Some approximations for the derived BFs are also proposed. A simulated data set is analyzed using the obtained results. Open problems are also mentioned. The findings of this paper may be used for assessing evidence in the available data in various fields such as reliability analyses of engineering systems and life testing experiments.

Keywords: Bayes Factor; Exponential model; Hypotheses testing; Likelihood ratio; Sequential order statistics

1 Introduction

Let $X_1,\ldots,X_n$ be independent and identically distributed (IID) random variables with a common distribution function (DF), say F, and abbreviated by $X_1,\ldots,X_n \overset{i.i.d.}{\sim} F$. Denote in magnitude order of $X_1,\ldots,X_n$ by $X_{1:n} \leq \cdots \leq X_{n:n}$, known as *order statistics* (OSs). The theory of OSs has been widely used in literature. For example, in system reliability analyses, lifetimes of r-out-of-n systems coincide to $X_{r:n}$, where $X_1,\ldots,X_n$ stand for component lifetimes. For more information, see Barlow and Proschan (1981) and David and Nagaraja (2003) and references therein. There are some generalizations of OSs such as *fractional order statistics* and *generalized order statistics*, which are useful for providing a framework to unify similar results in the related literature; see David and Nagaraja (2003) for more information. This paper deals with another unified concept,

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called the *sequential order statistics* (SOS), which has also a motivation in reliability analyses of engineering systems. Specifically, when the component lifetimes are IID, the OSs are suitable for describing r-out-of-n system lifetime. Here failing a component does not change the DFs of lifetimes of surviving components. Motivated by Cramer and Kamps (1996), the failure of a component may result in a higher load on the surviving components and hence causes the lifetime distributions change. More precisely, suppose that F_j, for $j=1,\ldots,n$, denotes the common DF of the component lifetimes when $n-j+1$ components are working. The components begin to work independently at time $t=0$ with the common DF F_1. When at time x_1, the first component failure occurs, the remaining $n-1$ components are working with the left truncated common DF F_2 at x_1. This process continues up to rth component failure and hence the system fails. The mentioned system is called *sequential r-out-of-n system* and its lifetime is then rth component failure time, denoted by $X^{\star}_{(r)}$. In the literature, $(X^{\star}_{(1)},\ldots,X^{\star}_{(n)})$ is called SOSs. Statistical inferences on the basis of SOSs have been studied in literature. For example, Bedbur (2010) obtained the uniformly most powerful unbiased test under a conditional proportional hazard rates (CPHR) model via a decision-theoretic approach. To describe the CPHR model, let $\bar{F}_j(t)=\bar{F}_0^{\alpha_j}(t)$, for $j=1,\ldots,r$, where $\bar{F}_0(t)=1-F_0(t)$ is a given baseline DF. In this case, the hazard rate function of the DF F_j, defined by $h_j(t)=f_j(t)/\bar{F}_j(t)$ for $t>0$ and $j=1,\ldots,n$, is proportional to the hazard rate function of the baseline DF F_0, that is, $h_j(t)=\alpha_j h_0(t)$. See also, Cramer and Kamps (2001a,b), Beutner and Kamps (2009), Schenk et al. (2011), Burkschat and Navarro (2011), Esmailian and Doostparast (2014), Hashempour and Doostparast (2017) and references therein. In this paper, we consider that the DF $F_0(t)$ is the exponential distribution, denoted by $Exp(\sigma)$, that is,

$$F_0(t;\sigma)=1-\exp\left\{-\left(\frac{t}{\sigma}\right)\right\},\qquad t>0,\quad \sigma>0. \tag{1}$$

The problem of hypotheses testing for exponential populations on the basis of $s(\geq 2)$ multiple and independent SOS samples under the CPHR model via a Bayesian approach is here studied. The available data are denoted by

$$\mathbf{x}=\begin{bmatrix} x_{11} & \ldots & x_{1r} \\ \vdots & \ddots & \vdots \\ x_{s1} & \ldots & x_{sr} \end{bmatrix}, \tag{2}$$

where the ith row of the matrix $\mathbf{x}$ in (2) stands for the SOS sample coming from the ith population $1\leq i\leq s$. In general, the likelihood function (LF) of the available data (2) reads

$$\begin{aligned} L(F_j^{[i]};1\leq i\leq s,1\leq j\leq r) &= \left(\frac{\Gamma(n+1)}{\Gamma(n-r+1)}\right)^s \prod_{i=1}^{s}\left(\prod_{j=1}^{r-1} f_j^{[i]}(x_{ij})\left(\frac{\bar{F}_j^{[i]}(x_{ij})}{\bar{F}_{j+1}^{[i]}(x_{ij})}\right)^{n-j}\right) \\ &\quad \times f_r^{[i]}(x_{ir})\bar{F}_r^{[i]}(x_{ir})^{n-r}, \end{aligned} \tag{3}$$

where $\bar{F}_j^{[i]}(x)=1-F_j^{[i]}(x)$, and $F_j^{[i]}$ calls for the common DF of the component lifetimes in the ith sequential r-out-of-n sample. For more details, refer to Hashempour and Doostparast (2017). Upon substituting (1) into (3), the LF (3) under the CPHR model reduces to

$$L(\sigma,\alpha;\mathbf{x})=\left(\frac{\Gamma(n+1)}{\Gamma(n-r+1)}\right)^s\left(\prod_{i=1}^{s}\frac{1}{\sigma_i}\right)^r \exp\left\{-\sum_{i=1}^{s}\sum_{j=1}^{r}\left(\frac{x_{ij}m_j}{\sigma_i}\right)\right\}, \tag{4}$$

where $\sigma = (\sigma_1, \ldots, \sigma_r)^T$, $\alpha = (\alpha_1, \ldots, \alpha_r)^T$ and $m_j = (n-j+1)\alpha_j - (n-j)\alpha_{j+1}$ for $j = 1, \ldots, r$, with convention $\alpha_{r+1} \equiv 0$. For the special case $\sigma_1 = \cdots = \sigma_s = \sigma$, the LF (4) simplifies to

$$L(\sigma, \alpha; \mathbf{x}) = \left(\frac{\Gamma(n+1)}{\Gamma(n-r+1)}\right)^s \left(\prod_{j=1}^{r} \alpha_j\right)^s \left(\frac{1}{\sigma}\right)^{sr} \exp\left\{-\left(\frac{\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j}{\sigma}\right)\right\}, \tag{5}$$

where σ is the common unknown mean of the baseline exponential DF in (1). In what follows, the following lemma is utilized.

Lemma 1. *Let $X^\star_{(1)}, \ldots, X^\star_{(n)}$ be SOSs under the CPHR model with the baseline $Exp(\sigma)$-distribution. Then, for $r = 1, \ldots, n$,*

$$\sum_{j=1}^{r} (n-j+1)\alpha_j D_{ij} = \sum_{j=1}^{r} X_{ij} m_j \sim gamma(r, \sigma), \tag{6}$$

where $D_{ij} = X_{ij} - X_{i,j-1}$, for $j = 1, \ldots, r$, $gamma(a,b)$ calls for the gamma distribution with density $f(x;a,b) = (\Gamma(a) b^a)^{-1} x^{a-1} \exp\{-(x/b)\}$, for $x > 0$,
and $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$ is the complete gamma function.

For more details, refer to Hashempour and Doostparast (2017). To the best of the anthers knowledge, Bayes factors (BFs) on the basis of SOSs has not been studied in the literature. This paper deals with this problem by emphasizing on SOSs coming for exponential baseline distribution under the CPHR model. So, the rest of this paper is organized as follow: In Section 2 , a review on BF and Bayes is given. A general form for BF is also derived for various hypotheses. In Section 3, BFs for SOS coming from exponential populations under the CPHR are provided. In Section 4, some approximations for the derived BF are proposed. These approximations are useful for numerical evaluations of the BFs specially in big data analyses. In Section 5, simulation studies based on SOS are provided. In Section 6, a real data set on failure times of aircraft components for a life test is analyzed. Section 7 concludes.

2 A review on BF

The BF is a Bayesian approach alternative to the frequentest one for comparing multiple candidate models based on the available data, say $\mathbf{x}$.

2.1 BF for simple hypothesis

Presence of nuisance parameters case the definition of the BF vague and complicated. Thus, in what follows, we consider two cases. As mentioned by Cowles (2013), in the Bayesian analysis when there are only two possible states of the world, M_1 and M_2 (or equivalently, two simple hypotheses H_1 and H_2), one may interest to compare the models with the prior probability $\pi(M_1) = 1 - \pi(M_2)$. Thus, the prior odd in favour of M_1(or H_1) is $\pi(M_1)/\pi(M_2)$ $(or\ \pi(H_1)/\pi(H_2))$. The posterior odd in favour of a model (or a hypothesis) is derived as the analogous ratio of posterior probabilities: $\pi(M_1|\mathbf{x})/\pi(M_2|\mathbf{x})\,(or\ \pi(H_1|\mathbf{x})/\pi(H_2|\mathbf{x}))$.

The BF in favour of a model or hypothesis is the ratio of the posterior odds to the prior odds. Thus, the BF in favour of M_1 versus M_2 is

$$BF = \frac{\pi(M_1|\mathbf{x})/\pi(M_2|\mathbf{x})}{\pi(M_1)/\pi(M_2)}. \tag{7}$$

The BF (7) is a summary of the evidence provided by the data $\mathbf{x}$ in favour of one scientific theory, represented by a statistical model, as opposed to another one. The BF usually is reported on the $\log_{10}$ scale. A review paper by Kass and Raftery (2012) recommends the interpretations of intervals of values of the BF as in Table 1.

Table 1: Interpretation of the strength of evidence

BF	Evidence against H_1
$0 < \text{BF} \le \frac{1}{10}$	Strong against
$\frac{1}{10} < \text{BF} \le \frac{1}{3}$	Substantial against
$\frac{1}{3} < \text{BF} \le 1$	Barely worth mentioning against
$1 < \text{BF} \le 3$	Barely worth mentioning
$3 < \text{BF} \le 10$	Substantial
$10 < \text{BF} < \infty$	Strong

Let $f(\mathbf{y}|M_i)$, $(i = 1, 2)$ stand for the probability density function (PDF) of $\mathbf{y}$ given the ith model. Then, the BF (7) for comparing to models M_1 and M_2, or equivalently for testing H_1. The model M_1 is correct against the alternative H_2. The model M_2 is correct, is simplified as

$$BF = \frac{\frac{f(\mathbf{y}|M_1)\Pi(M_1)}{f(\mathbf{y}|M_2)\Pi(M_2)}}{\frac{\Pi(M_1)}{\Pi(M_2)}} = \frac{f(\mathbf{y}|M_1)}{f(\mathbf{y}|M_2)}. \tag{8}$$

Equation (8) means that, the BF is the ratio of the likelihoods under the two simple hypotheses. In other words, it is the evidence contained in the data alone (uninfluenced by the prior) in favour of one model over the other.

Example 1. On the basis of the observed data $\underline{x}$ in (2) and under the CPHR model, described in the preceding section with the baseline $Exp(\sigma)$-distribution, consider the problem of hypotheses testing

$$H_1 : \sigma = \sigma_1 \quad v.s \quad H_2 : \sigma = \sigma_2, \tag{9}$$

where σ_1 and σ_2 are known positive constants and $0 < \sigma_1 < \sigma_2$. Equations (5) and (8) get

$$\begin{aligned} BF = \frac{L(\sigma_1; \underline{x})}{L(\sigma_2; \underline{x})} &= \left(\frac{\sigma_2}{\sigma_1}\right)^{sr} \exp\left\{-\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j\right\} \\ &= \left(\frac{\sigma_2}{\sigma_1}\right)^{sr} \exp\left\{-\left(\frac{1}{\sigma_1} - \frac{1}{\sigma_2}\right)\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij}\right\}. \end{aligned} \tag{10}$$

Note that, BF (10) in the simple versus simple case is the weight of evidence contained in the data alone in favour of M_1 versus M_2. Thus, it ignores any information provided by the priors. For more details, see Hashempour and Doostparast (2016, 2017).

2.2 BF for composite hypotheses

In presence of unknown parameters, say θ, the BF given by (7) is not useful. For these cases, the marginal likelihoods may be used. The numeric value of a marginal likelihood is determined by the data and the entire Bayesian model (the form of the likelihood and all levels of priors). To do this, suppose we wish to compare two families of models, denoted by $\mathscr{M}_1$ and $\mathscr{M}_2$, based on the observed data $\mathbf{y}$. The two families may have different likelihoods, different numbers of unknown parameters, and so on. The BF in the general case is the ratio of the marginal likelihoods under the two candidate families of models. Let $\theta_\mathbf{i}$ $(i = 1, 2)$ denote parameters for the family $\mathscr{M}_i$. The marginal likelihoods under the family $\mathscr{M}_i$ is defined by

$$P(\mathbf{y}|\mathscr{M}_i) = \int P(\mathbf{y}|\theta_i)P(\theta_i|\mathscr{M}_i)d\theta_i. \tag{11}$$

Therefore, (8) motivates us to define the BF as

$$BF = \frac{P(\mathbf{y}|\mathscr{M}_1)}{P(\mathbf{y}|\mathscr{M}_2)}. \tag{12}$$

The suggested BF (12) cannot be interpreted as the evidence in the data alone, since clearly the priors affect each marginal likelihood and therefore the BF itself. For more details, refer to Lewis and Raftery (1997) and Klauer et al. (2024).

3 SOS-based BF for exponential populations

In general, we are interested in comparing composite hypotheses $H_1 : \sigma \in \Omega_1$ against the alternative $H_2 : \sigma \in \Omega_2$ where Ω is the parameter space, $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$. Here "$\emptyset$" stands for the empty set. Suppose that $\pi(\sigma)$ is the prior density on the parameter space Ω. To derive the BF on the basis of data x in (2), assume that the parameter vector α in (4) is known, and it is suggested to consider the conjugate prior distribution for the scale parameters σ as $\sigma \sim IG(a, b)$, which is the inverse gamma distribution with shape and scale parameters a and b, respectively. The PDF σ is defined as follows:

$$\pi(\sigma) = \frac{b^a}{\Gamma(a)}\sigma^{-(a+1)} \exp\left\{-\left(\frac{b}{\sigma}\right)\right\}, \qquad \sigma > 0, \quad a > 0, \; b > 0. \tag{13}$$

Equations (4) and (13) imply the posterior distribution of σ given $\underline{x}$ as

$$\sigma \mid \underline{x} \sim IG\left(a_i + r, \sum_{j=1}^{r} x_{ij} m_j + b_i\right), \qquad i = 1, \ldots, s. \tag{14}$$

Remark 1. *Under the squared error loss (SEL) function, that is, $L(\theta, \delta) = (\delta - \theta)^2$, where θ is the parameter of interest and δ is an estimate of θ, the Bayes estimate of the parameter θ is the posterior mean. Thus, the Bayes estimate of σ is*

$$\hat{\sigma}_B = \frac{r\hat{\sigma} + b}{a + r - 1}, \tag{15}$$

where $\hat{\sigma}$ is the ML estimate of σ given by $\hat{\sigma}=\sum_{j=1}^{r} x_{ij}m_j/r=\sum_{j=1}^{r}(n-j+1)\alpha_j D_{ij}/r$. Note that the Bayes estimate (15) is a weighted mean of the mean of the prior (13) and the ML estimate above; that is, $\hat{\sigma}_B = E(\sigma)w+(1-w)\hat{\sigma}$, where $w=(a-1)/(a+r-1)$. For $r=n$ and $\alpha_1=\cdots=\alpha_n=1$, we have $\hat{\sigma}_n=\sum_{j=1}^{n} x_{ij}/n$ and $\hat{\sigma}_B=(\sum_{j=1}^{n} x_{ij}+b)/(a+n-1)$, which are, respectively, the well-known ML and the Bayes estimates of the exponential parameters on the basis of a random sample of size n; see, for example, Lawless (2003) and Hashempour and Doostparast (2016).

Proposition 1. *Let $\pi_1(\sigma)$ and $\pi_2(\sigma)$ be two proper densities over Ω_1 and Ω_2, respectively. Then the BF for $H_1:\sigma\in\Omega_1$ against $H_2:\sigma\in\Omega_2$ is*

$$BF_{1,2}=\frac{\Gamma(a_2)\Gamma(a_1^*)b_1^{a_1}(b_2^*)^{a_2^*}}{\Gamma(a_1)\Gamma(a_2^*)b_2^{a_2}(b_1^*)^{a_1^*}}\left(\frac{P(\chi_{2a_1^*}\in 2b_1^*.\Omega_1^{[-1]})}{P(\chi_{2a_2^*}\in 2b_2^*.\Omega_2^{[-1]})}\right)\left(\frac{P(\chi_{2a_2}\in 2b_2.\Omega_2^{[-1]})}{P(\chi_{2a_1}\in 2b_1.\Omega_1^{[-1]})}\right), \tag{16}$$

where $2b_i^*.\Omega_i^{[-1]}=\{2b_i^*\theta^{-1}:\theta\in\Omega\}$, $2b_i.\Omega_i^{[-1]}=\{2b_i\theta^{-1}:\theta\in\Omega\}$, $b_i^*=b_i+\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j$ and $a_i^*=a_i+sr$, for $i=1,2$ and χ_ν stands for the chi-square distribution with ν degrees of freedom.

Proof. From (12), (13), and (14), the BF of H_1 against H_2 is

$$\begin{aligned}
BF_{1,2}&=\left(\frac{\int_{\Omega_1}L(\sigma|\underline{\mathbf{x}})\pi_1(\sigma)d\sigma}{\int_{\Omega_2}L(\sigma|\underline{\mathbf{x}})\pi_2(\sigma)d\sigma}\right)\left(\frac{\int_{\Omega_2}\pi_2(\sigma)d\sigma}{\int_{\Omega_1}\pi_1(\sigma)d\sigma}\right)\\
&=\frac{\int_{\Omega_1}A^s\left(\prod_{j=1}^{r}\alpha_j\right)^s\left(\prod_{i=1}^{s}\frac{1}{\sigma}\right)^r\exp\left\{-\sum_{i=1}^{s}\sum_{j=1}^{r}\left(\frac{x_{ij}m_j}{\sigma_i}\right)\right\}\frac{b_1^{a_1}}{\Gamma(a_1)}\sigma^{-(a_1+1)}}{\int_{\Omega_2}A^s\left(\prod_{j=1}^{r}\alpha_j\right)^s\left(\prod_{i=1}^{s}\frac{1}{\sigma}\right)^r\exp\left\{-\sum_{i=1}^{s}\sum_{j=1}^{r}\left(\frac{x_{ij}m_j}{\sigma_i}\right)\right\}\frac{b_2^{a_2}}{\Gamma(a_2)}\sigma^{-(a_2+1)}}\\
&\quad\times\frac{\exp\left\{-\left(\frac{b_2}{\sigma}\right)\right\}d\sigma\int_{\Omega_2}\frac{b_2^{a_2}}{\Gamma(a_2)}\sigma^{-(a_2+1)}\exp\left\{-\left(\frac{b_2}{\sigma}\right)\right\}d\sigma}{\exp\left\{-\left(\frac{b_1}{\sigma}\right)\right\}d\sigma\int_{\Omega_1}\frac{b_1^{a_1}}{\Gamma(a_1)}\sigma^{-(a_1+1)}\exp\left\{-\left(\frac{b_1}{\sigma}\right)\right\}d\sigma}\\
&=\frac{\Gamma(a_2)b_1^{a_1}}{\Gamma(a_1)b_2^{a_2}}\frac{\int_{\Omega_1}\sigma^{-(a_1+sr+1)}\exp\{-\frac{1}{\sigma}\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j+b_1\right)\}d\sigma}{\int_{\Omega_2}\sigma^{-(a_2+sr+1)}\exp\{-\frac{1}{\sigma}\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j+b_2\right)\}d\sigma}\\
&\quad\times\frac{\int_{\Omega_2}\frac{b_2^{a_2}}{\Gamma(a_2)}\sigma^{-(a_2+1)}\exp\left\{-\left(\frac{b_2}{\sigma}\right)\right\}d\sigma}{\int_{\Omega_1}\frac{b_1^{a_1}}{\Gamma(a_1)}\sigma^{-(a_1+1)}\exp\left\{-\left(\frac{b_1}{\sigma}\right)\right\}d\sigma}\\
&=\frac{\Gamma(a_2)\Gamma(a_1+sr)b_1^{a_1}\left(b_2+\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right)^{a_2+sr}}{\Gamma(a_1)\Gamma(a_2+sr)b_2^{a_2}\left(b_1+\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right)^{a_1+sr}}\frac{P(\chi_{2(a_1+sr)}\in\Omega_1)}{P(\chi_{2(a_2+sr)}\in\Omega_2)}\times\frac{P(\chi_{2a_1}\in\Omega_2)}{P(\chi_{2a_2}\in\Omega_1)}\\
&=\frac{\Gamma(a_2)\Gamma(a_1^*)b_1^{a_1}(b_2^*)^{a_2^*}}{\Gamma(a_1)\Gamma(a_2^*)b_2^{a_2}(b_1^*)^{a_1^*}}\left(\frac{P(\chi_{2a_1^*}\in 2b_1^*.\Omega_1^{[-1]})}{P(\chi_{2a_2^*}\in 2b_2^*.\Omega_2^{[-1]})}\right)\left(\frac{P(\chi_{2a_2}\in 2b_2.\Omega_2^{[-1]})}{P(\chi_{2a_1}\in 2b_1.\Omega_1^{[-1]})}\right).\\
&=A\left(\frac{P(\chi_{2a_1^*}\in 2b_1^*.\Omega_1^{[-1]})}{P(\chi_{2a_2^*}\in 2b_2^*.\Omega_2^{[-1]})}\right)\left(\frac{P(\chi_{2a_2}\in 2b_2.\Omega_2^{[-1]})}{P(\chi_{2a_1}\in 2b_1.\Omega_1^{[-1]})}\right),
\end{aligned}$$

where $A=\frac{\Gamma(a_2)\Gamma(a_1^*)b_1^{a_1}(b_2^*)^{a_2^*}}{\Gamma(a_1)\Gamma(a_2^*)b_2^{a_2}\left(b_1^*\right)^{a_1^*}}$. □

In what follows, for the proper prior $\pi_i(\sigma)(i=1,2)$ in Proposition 1, the truncated inverse gamma distributions to the parameter of spaces is assumed, that is,

$$\pi_i(\sigma) = \frac{b_i^{a_i}}{\Gamma(a_i)} \sigma^{-(a_i+1)} \exp\left\{-\left(\frac{b_i}{\sigma}\right)\right\} \times \frac{1}{\int_{\Omega_i} IG(a_i, b_i) d\sigma}, \quad i = 1, 2, \ \sigma \in \Omega_i.$$

A general form for the SOS-based BF in (16) is derived in terms of the chi-square DF. For some common hypotheses, the proposed BF in (16) may be simplified. Similar to Lehmann and Romano (2005, Ch 4.), the following hypotheses are considered and the corresponding simplified BFs are displayed in Table 2:

$$H_3 : \sigma \leq \sigma_0 \text{ v.s } H_4 : \sigma > \sigma_0 \tag{17}$$

$$H_5 : \sigma \geq \sigma_0 \text{ v.s } H_6 : \sigma < \sigma_0 \tag{18}$$

$$H_7 : \sigma = \sigma_0 \text{ v.s } H_8 : \sigma > \sigma_0 \tag{19}$$

$$H_9 : \sigma = \sigma_0 \text{ v.s } H_{10} : \sigma < \sigma_0 \tag{20}$$

$$H_{11} : \sigma_1 \leq \sigma \leq \sigma_2 \text{ v.s } H_{12} : \sigma > \sigma_2 \text{ or } \sigma < \sigma_1 \tag{21}$$

$$H_{13} : \sigma \geq \sigma_2 \text{ or } \sigma \leq \sigma_1 \text{ v.s } H_{14} : \sigma_1 < \sigma < \sigma_2 \tag{22}$$

$$H_{15} : \sigma = \sigma_0 \text{ v.s } H_{16} : \sigma \neq \sigma_0. \tag{23}$$

Here, σ_0, σ_1, and σ_2 are known positive constants and $\sigma_1 < \sigma_2$. In Table 2, we have

$$A = \left(\Gamma(a_2)\Gamma(a_1^*) b_1^{a_1} (b_2^*)^{a_2^*}\right) / \left(\Gamma(a_1)\Gamma(a_2^*) b_2^{a_2} (b_1^*)^{a_1^*}\right)$$

and

$$B = \left(\Gamma(a_2)(b_2^*)^{a_2^*}\right) / (\Gamma(a_2^*) b_2^{a_2}),$$

and F_{χ_ν} stands for the DF of the chi-square distribution with ν degrees of freedom.

Lemma 2 (Johnson et al. (1994))**.** *For $t > 0$,*

$$F_{\chi_\nu}(t) = 1 - \exp\left\{-\frac{t}{2}\right\} \sum_{i=0}^{\nu-1} \frac{(\frac{t}{2})^i}{i!}. \tag{24}$$

Lemma 2 gives an alternative method for the expression of the BFs associated with the hypotheses (17)-(23); see Table 3.

4 Approximate BF

The BFs in (16) and Table 2 involve the DF of the chi-square distribution. In this section, some approximations for the BF defined by (16) are proposed, which may be useful for numerical evaluations indeed in big data analyses. To do this, some lemmas are given. The first lemma is based on the cumulative distribution function (CDF) of the standard normal CDF, that is,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\{-u^2/2\} du. \tag{25}$$

Table 2: BFs

$BF_{3,4}=A\left(\frac{1-F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_0})}{1-F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_0})}\right)\left(\frac{F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_0})}{F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_0})}\right)$
$BF_{5,6}=A\left(\frac{1-F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_0})}{1-F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_0})}\right)\left(\frac{F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_0})}{F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_0})}\right)$
$BF_{7,8}=\frac{B}{\sigma_0^{sr}}\exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right)\right\}\left(\frac{F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_0})}{F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_0})}\right)$
$BF_{9,10}=\frac{B}{\sigma_0^{sr}}\exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right)\right\}\left(\frac{1-F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_0})}{1-F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_0})}\right)$
$BF_{11,12}=A\left(\frac{F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_1})-F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_2})}{F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_1})-F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_2})}\right)\left(\frac{1-F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_1})+F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_2})}{1-F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_1})+F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_2})}\right)$
$BF_{13,14}=A\left(\frac{1-F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_1})+F_{\chi_{2a_1^*}}(\frac{2b_1^*}{\sigma_2})}{1-F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_1})+F_{\chi_{2a_1}}(\frac{2b_1}{\sigma_2})}\right)\left(\frac{F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_1})-F_{\chi_{2a_2}}(\frac{2b_2}{\sigma_2})}{F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_1})-F_{\chi_{2a_2^*}}(\frac{2b_2^*}{\sigma_2})}\right)$
$BF_{15,16}=\frac{B}{\sigma_0^{sr}}\exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r}x_{ij}m_j\right)\right\}$

Lemma 3 (Johnson et al. (1994), page 426)**.** *As $\nu\to+\infty$, we have for all $t>0$*

(I) $F_{\chi_\nu}(t)\approx\Phi\left(\frac{t-\nu}{\sqrt{2\nu}}\right)$,

(II) $F_{\chi_\nu}(t)\approx\Phi\left(\sqrt{2t}-\sqrt{2\nu-1}\right)$.

In Lemma 3, the second approximation is better than the first one; see, for example, Johnson et al. (1994). Meanwhile, we provide some approximations for BF with both of them. So, Lemma 3 gives

$$BF_{11,12}^{[I]}=A\left(\frac{\Phi\left(\frac{\frac{2b_1^*}{\sigma_1}-2a_1^*}{\sqrt{4a_1^*}}\right)-\Phi\left(\frac{\frac{2b_1^*}{\sigma_2}-2a_1^*}{\sqrt{4a_1^*}}\right)}{\Phi\left(\frac{\frac{2b_1}{\sigma_1}-2a_1}{\sqrt{4a_1}}\right)-\Phi\left(\frac{\frac{2b_1}{\sigma_2}-2a_1}{\sqrt{4a_1}}\right)}\right)\left(\frac{1-\Phi\left(\frac{\frac{2b_2}{\sigma_1}-2a_2}{\sqrt{4a_2}}\right)+\Phi\left(\frac{\frac{2b_2}{\sigma_2}-2a_2}{\sqrt{4a_2}}\right)}{1-\Phi\left(\frac{\frac{2b_2^*}{\sigma_1}-2a_2^*}{\sqrt{4a_2^*}}\right)+\Phi\left(\frac{\frac{2b_2^*}{\sigma_2}-2a_2^*}{\sqrt{4a_2^*}}\right)}\right), \tag{26}$$

$$\begin{aligned}BF_{11,12}^{[II]} &= A\left(\frac{\Phi\left(\sqrt{\frac{4b_1^*}{\sigma_1}}-\sqrt{4a_1^*-1}\right)-\Phi\left(\sqrt{\frac{4b_1^*}{\sigma_2}}-\sqrt{4a_1^*-1}\right)}{\Phi\left(\sqrt{\frac{4b_1}{\sigma_1}}-\sqrt{4a_1-1}\right)-\Phi\left(\sqrt{\frac{4b_1}{\sigma_2}}-\sqrt{4a_1-1}\right)}\right)\\ &\quad\times\left(\frac{1-\Phi\left(\sqrt{\frac{4b_2}{\sigma_1}}-\sqrt{4a_2-1}\right)+\Phi\left(\sqrt{\frac{4b_2}{\sigma_2}}-\sqrt{4a_2-1}\right)}{1-\Phi\left(\sqrt{\frac{4b_2^*}{\sigma_1}}-\sqrt{4a_2^*-1}\right)+\Phi\left(\sqrt{\frac{4b_2^*}{\sigma_2}}-\sqrt{4a_2^*-1}\right)}\right),\end{aligned} \tag{27}$$

Table 3: BFs based on Lemma 2

$BF_{3,4} = A \left(\frac{\exp\left\{-\frac{b_1^*}{\sigma_0}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_0})^i}{i!}}{\exp\left\{-\frac{b_1}{\sigma_0}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_0})^i}{i!}} \right) \left(\frac{1-\exp\left\{-\frac{b_2}{\sigma_0}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_0})^i}{i!}}{1-\exp\left\{-\frac{b_2^*}{\sigma_0}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_0})^i}{i!}} \right)$
$BF_{5,6} = A \left(\frac{\exp\left\{-\frac{b_2}{\sigma_0}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_0})^i}{i!}}{\exp\left\{-\frac{b_2^*}{\sigma_0}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_0})^i}{i!}} \right) \left(\frac{1-\exp\left\{-\frac{b_1^*}{\sigma}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_0})^i}{i!}}{1-\exp\left\{-\frac{b_1}{\sigma_0}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_0})^i}{i!}} \right)$
$BF_{7,8} = \frac{B}{\sigma_0^{sr}} \exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j\right)\right\} \left(\frac{1-\exp\left\{-\frac{b_2}{\sigma_0}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_0})^i}{i!}}{1-\exp\left\{-\frac{b_2^*}{\sigma_0}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_0})^i}{i!}} \right)$
$BF_{9,10} = \frac{B}{\sigma_0^{sr}} \exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j\right)\right\} \left(\frac{\exp\left\{-\frac{b_2}{\sigma_0}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_0})^i}{i!}}{\exp\left\{-\frac{b_2^*}{\sigma_0}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_0})^i}{i!}} \right)$
$BF_{11,12} = A \left(\frac{\exp\left\{-\frac{b_1^*}{\sigma_1}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_1^*}{\sigma_2}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_2})^i}{i!}}{\exp\left\{-\frac{b_1}{\sigma_1}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_1}{\sigma_2}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_2})^i}{i!}} \right)$ $\times \left(\frac{1+\exp\left\{-\frac{b_2}{\sigma_1}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_2}{\sigma_2}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_2})^i}{i!}}{1+\exp\left\{-\frac{b_2^*}{\sigma_1}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_2^*}{\sigma_2}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_2})^i}{i!}} \right)$
$BF_{13,14} = A \left(\frac{1+\exp\left\{-\frac{b_1^*}{\sigma_1}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_1^*}{\sigma_2}\right\} \sum_{i=0}^{a_1^*-1} \frac{(\frac{b_1^*}{\sigma_1})^i}{i!}}{1+\exp\left\{-\frac{b_1}{\sigma_1}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_1}{\sigma_2}\right\} \sum_{i=0}^{a_1-1} \frac{(\frac{b_1}{\sigma_2})^i}{i!}} \right)$ $\times \left(\frac{\exp\left\{-\frac{b_2}{\sigma_1}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_2}{\sigma_2}\right\} \sum_{i=0}^{a_2-1} \frac{(\frac{b_2}{\sigma_2})^i}{i!}}{\exp\left\{-\frac{b_2^*}{\sigma_1}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_1})^i}{i!} - \exp\left\{-\frac{b_2^*}{\sigma_2}\right\} \sum_{i=0}^{a_2^*-1} \frac{(\frac{b_2^*}{\sigma_2})^i}{i!}} \right)$
$BF_{15,16} = \frac{B}{\sigma_0^{sr}} \exp\left\{-\frac{1}{\sigma_0}\left(\sum_{i=1}^{s}\sum_{j=1}^{r} x_{ij} m_j\right)\right\}$

$$BF_{3,4}^{[I]} = A \left(\frac{1-\Phi\left(\frac{\frac{2b_1^*}{\sigma_0}-2a_1^*}{\sqrt{4a_1^*}}\right)}{1-\Phi\left(\frac{\frac{2b_1}{\sigma_0}-2a_1}{\sqrt{4a_1}}\right)} \right) \left(\frac{\Phi\left(\frac{\frac{2b_2}{\sigma_0}-2a_2}{\sqrt{4a_2}}\right)}{\Phi\left(\frac{\frac{2b_2^*}{\sigma_0}-2a_2^*}{\sqrt{4a_2^*}}\right)} \right), \tag{28}$$

$$BF_{3,4}^{[II]} = A \left(\frac{1-\Phi\left(\sqrt{\frac{4b_1^*}{\sigma_0}} - \sqrt{4a_1^*-1}\right)}{1-\Phi\left(\sqrt{\frac{4b_1}{\sigma_0}} - \sqrt{4a_1-1}\right)} \right) \left(\frac{\Phi\left(\sqrt{\frac{4b_2}{\sigma_0}} - \sqrt{4a_2-1}\right)}{\Phi\left(\sqrt{\frac{4b_2^*}{\sigma_0}} - \sqrt{4a_2^*-1}\right)} \right). \tag{29}$$

The next lemma presents another approximation the CDF of the χ_ν distribution base on an infinite series. So, the CDF can be approximated by computing the summation for some finite elements.

Lemma 4 (Johnson et al. (1994)). *For $x > 0$,*

$$F_{\chi_\nu}(t) = \frac{2(2t)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})} \sum_{i=0}^{\infty} (-1)^i \frac{t^i}{(\nu+2i)2^i i!}. \tag{30}$$

The proposed BFs can be approximated by Lemma 4. For example,

$$\begin{aligned} BF_{11,12} &= A \frac{\frac{2^{a_1^*}}{\Gamma(a_1^*)} \sum_{i=0}^{N} \left\{ \frac{(-2)^i (2b_1^*)^{i+a_1^*}}{i!\left(a_1^*+i\right)} \left(\left(\frac{1}{\sigma_1}\right)^{i+a_1^*} - \left(\frac{1}{\sigma_2}\right)^{i+a_1^*} \right) \right\}}{\frac{2^{a_1}}{\Gamma(a_1)} \sum_{i=0}^{N} \frac{(-2)^i (2b_1)^{i+a_1}}{i!(a_1+i)} \left(\left(\frac{1}{\sigma_1}\right)^{i+a_1} - \left(\frac{1}{\sigma_2}\right)^{i+a_1} \right)} \\ &\quad \times \frac{1 + \frac{2^{a_2}}{\Gamma(a_2)} \sum_{i=0}^{N} \frac{(-2)^i (2b_2)^{i+a_2}}{i!(a_2+i)} \left(\left(\frac{1}{\sigma_2}\right)^{i+a_2} - \left(\frac{1}{\sigma_1}\right)^{i+a_2} \right)}{1 + \frac{2^{a_2^*}}{\Gamma(a_2^*)} \sum_{i=0}^{N} \frac{(-2)^i \left(2b_2^*\right)^{i+a_2^*}}{i!\left(a_2^*+i\right)} \left(\left(\frac{1}{\sigma_2}\right)^{i+a_2^*} - \left(\frac{1}{\sigma_1}\right)^{i+a_2^*} \right)}, \end{aligned} \tag{31}$$

where N is a large number. Other approximations methods such as the Laplace method can be used to approximate the F-distribution function; see, for example, Johnson et al. (1994). A similar approach then is used to obtain an approximate BF based on SOSs.

5 Simulation studies

To examine the accuracy of the proposed BF, we performed a Monte Carlo simulation study in the well-known statistical software R. For generating an SOS-sample from the exponential population with $\sigma = 1$ under the CPHR model, an algorithm proposed by Cramer and Kamps (1996) was performed. Here, we considered the hypothesis $H_{11} : \sigma_1 \le \sigma \le \sigma_2$ v.s $H_{12} : \sigma > \sigma_2$ or $\sigma < \sigma_1$.

In Table 4 and Figure 1, the mean of the BFs based on 10^4 iterations for some selected reduces of n and r are displayed. Appr 1 and Appr 2 stand for the approximations based on Lemma (3) and (4), respectively.

Table 5 and Figure 2 represent the mean absolute among approximate and exact BFs.

Empirical results are

- increasing r more effective than increasing the copy s;
- approximations tend to the actual value as r/n increasing;
- Appr 1 dominates Appr 2;
- As $n \longrightarrow \infty$ and r/n goes to unify, the BF determines successfully the correct hypothesis.

Table 4: Exact values and the corresponding approximates for the BF on the basis of a SOS-sample from the exponential population under the CPHR model for some selected values of n and r.

	Exact	Appr1	Appr2
3	1.9843	2.0852	2.6624
4	1.5754	1.6474	2.0768
5	1.2117	1.2621	1.5760
6	0.9860	1.0200	1.2533
7	0.6417	0.6731	0.8562
8	0.5612	0.5880	0.7453
9	0.3924	0.4116	0.5234
10	0.3719	0.4053	0.4997
11	0.2948	0.3078	0.3863
12	0.2341	0.2442	0.3054

(a) $n = 20,\ r = 10$

	Exact	Appr1	Appr2
3	1.3629	1.4212	1.7790
4	0.8295	0.8679	1.0981
5	0.6116	0.6389	0.8052
6	0.4283	0.4472	0.5619
7	0.2965	0.3103	0.3921
8	0.2345	0.2451	0.3080
9	0.1707	0.1780	0.2224
10	0.1221	0.1271	0.1579
11	0.1067	0.1109	0.1371
12	0.0791	0.0821	0.1010

(b) $n = 20,\ r = 15$

	Exact	Appr1	Appr2
3	2.5126	2.6767	3.5636
4	2.4532	2.5998	3.4039
5	2.3153	2.4416	3.1484
6	1.9026	2.0020	2.5661
7	1.7864	1.8722	2.3728
8	1.5225	1.5929	2.0113
9	1.3309	1.3891	1.7427
10	1.1886	1.2388	1.5503
11	0.9422	0.9870	1.2512
12	0.8756	0.9137	1.1486

(c) $n = 20,\ r = 5$

	Exact	Appr1	Appr2
3	2.4105	2.5696	3.4268
4	2.4171	2.5623	3.3578
5	2.1453	2.2650	2.9317
6	1.8881	1.9863	2.5442
7	1.7795	1.8660	2.3694
8	1.5757	1.6483	2.0789
9	1.3484	1.4092	1.7730
10	1.1382	1.1905	1.5024
11	1.0839	1.1308	1.4183
12	0.9575	0.9928	1.2282

(d) $n = 10,\ r = 5$

6 Aircraft data set

To demonstrate the results obtained in the preceding sections, we present an illustrative example. Smith (2002) gave failure times of aircraft components for a life-test, originally due to Mann and Fertig (1973). In the test, $n = 13$ components were placed in a Type-II censored life test in which the failure times of first 10 components to fail were observed (in hours) as $0.22, 0.50, 0.88, 1.00, 1.32, 1.33, 1.54, 1.76, 2.50, 3.00$. Following Hashempour et al. (2019), it is assumed that the lifetimes of the components are IID with an exponential distribution. We considered two simple hypothesis tests based on the ML estimate of the σ in (15). Also, we ran the SOS example for $r = 3,4$ and $s = 3,4,5$. The BF is approximated using (31) for the failure time of aircraft components. The results on Table 6 show that as r or s increases, BF determines the correct hypothesis more successfully.

Table 5: Absolute errors of the approximations in Table 4 for the BF on the basis of a SOS-sample from the exponential population under the CPHR model for some selected values of n and r.

	Appr1	Appr2
3	0.1025	0.6823
4	0.0743	0.5052
5	0.0586	0.3724
6	0.0496	0.2989
7	0.0327	0.2175
8	0.0279	0.1867
9	0.0203	0.1334
10	0.0147	0.1297
11	0.0141	0.0936
12	0.0109	0.0732

(a) $n = 20$, $r = 10$

	Appr1	Appr2
3	0.0631	0.4197
4	0.0420	0.2717
5	0.0319	0.2030
6	0.0206	0.1361
7	0.0148	0.0978
8	0.0113	0.0754
9	0.0080	0.0534
10	0.0056	0.0373
11	0.0048	0.0318
12	0.0035	0.0231

(b) $n = 20$, $r = 15$

	Appr1	Appr2
3	0.1661	1.0566
4	0.1484	0.9558
5	0.1280	0.8377
6	0.1010	0.6677
7	0.0873	0.5903
8	0.0727	0.4928
9	0.0632	0.4154
10	0.0565	0.3698
11	0.0493	0.3151
12	0.0436	0.2808

(c) $n = 20$, $r = 5$

	Appr1	Appr2
3	0.1610	1.0216
4	0.1470	0.9456
5	0.1215	0.7911
6	0.0998	0.6605
7	0.0881	0.5940
8	0.0749	0.5070
9	0.0653	0.4283
10	0.0581	0.3716
11	0.0536	0.3444
12	0.0469	0.2913

(d) $n = 10$, $r = 5$

7 Conclusion

This paper focused on calculating BFs using sequential order statistics arising from homogeneous exponential DFs. Also various approximations for these BFs were proposed. A simulation study was conducted, and real data set was illustrated. The discoveries presented in this paper have practical applications in evaluating evidence in various domains, including reliability analysis of engineering systems and life testing experiments.

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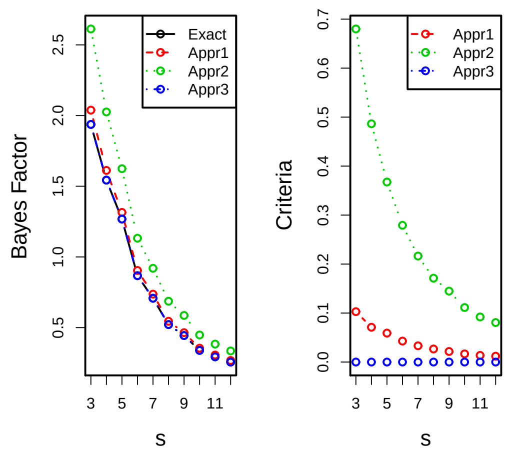
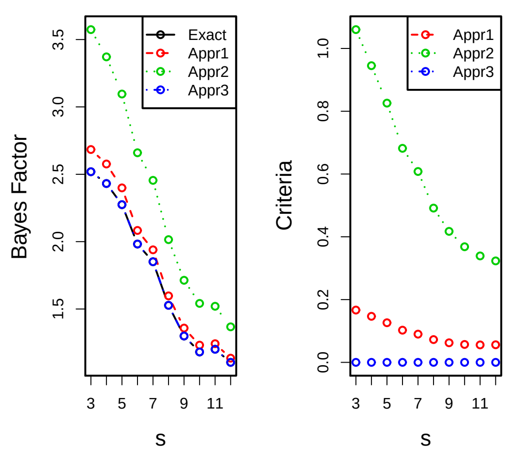


(a) $n = 20,\ r = 10$



(b) $n = 20,\ r = 5$

Figure 1: BF and criteria.

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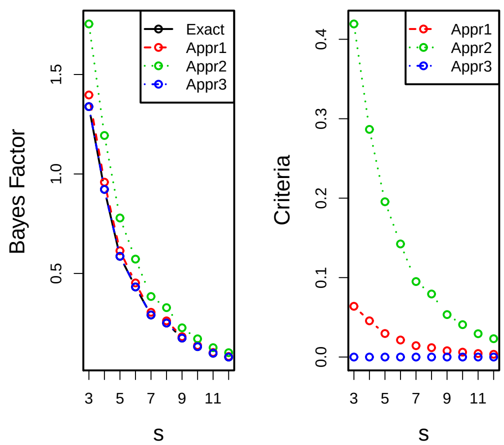
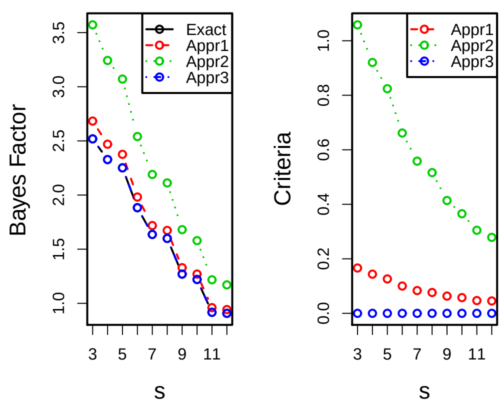


(a) $n = 20,\ r = 15$



(b) $n = 10,\ r = 5$

Figure 2: BF and criteria.

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Table 6: Approximations of the BF for failure times of aircraft components.

H_1 vs H_2	r	s	BF
$H_1 : \sigma = 5$	3	3	2.112
$H_2 : \sigma = 6$		4	2.649
		5	6.235
	4	3	4.886
		4	13.537
		5	16.942
$H_1 : \sigma = 5$	3	3	10.784
$H_2 : \sigma = 7$		4	22.507
		5	81.136
	4	3	49.640
		4	301.139
		5	463.789

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