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Wiener index and saturated nodes of random exponential recursive ternary trees

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Abstract. An exponential recursive ternary tree (ERTT) is defined by the property that each node has three possible positions, referred to as external nodes, to which a child may be attached. At each growth step, every external node independently becomes a leaf with probability p, or remains external with probability $1-p$. In this note, we study two quantities associated with ERTTs: the external Wiener index, defined as the sum of distances over all pairs of external nodes, and the number of saturated nodes, i.e., nodes that have three children. For these trees, we derive the expected value and the second moment of both the number of saturated nodes and the Wiener index. Subsequently, applying the martingale convergence theorem, we establish almost sure convergence and convergence in quadratic mean for a suitably scaled external Wiener index, which reflects the typical distance between two randomly chosen external nodes. Finally, using the contraction method, we prove convergence in distribution for a scaled version of the number of saturated nodes.

Keywords: Contraction method; L^2-norm; Martingale convergence theorem; Wasserstein metric; Wiener index.

1 Introduction and setting

A *random recursive tree* of size n is built sequentially: starting with a root, each new node attaches uniformly at random to one of the existing nodes Smythe and Mahmoud (1995). For only a few of variations of recursive trees, see Javanian and Vahidi-Asl (2003), Javanian and Vahidi-Asl (2006), Javanian and Vahidi-Asl (2012), Javanian (2013), Javanian (2017) and Zhang (2018). These trees have been widely studied and applied in various fields Chan et al. (2003), Najock and Heyde (1982), Moon (1974).

Since only one node is added at each step in the growth of a recursive tree, then these slow-growing models cannot be suitable for fast-growing phenomena such as the growing the Twitter and the spreading of infectious diseases. In Mahmoud (2022), it has been proposed a fast-growing recursive tree model, i.e. the *exponential recursive tree* (ERT), where every node independently attracts a new node with probability p, or fails to do at each step in the growth of the tree. The *size* (the number of nodes), the *leaves* (the nodes with no children), the *depth* of a node v (the distance, i.e. the number of edges, of node v from tree's root node) and the *total path length* (the sum of depth of all nodes) have been studied for ERTs by Aguech et al. (2021).

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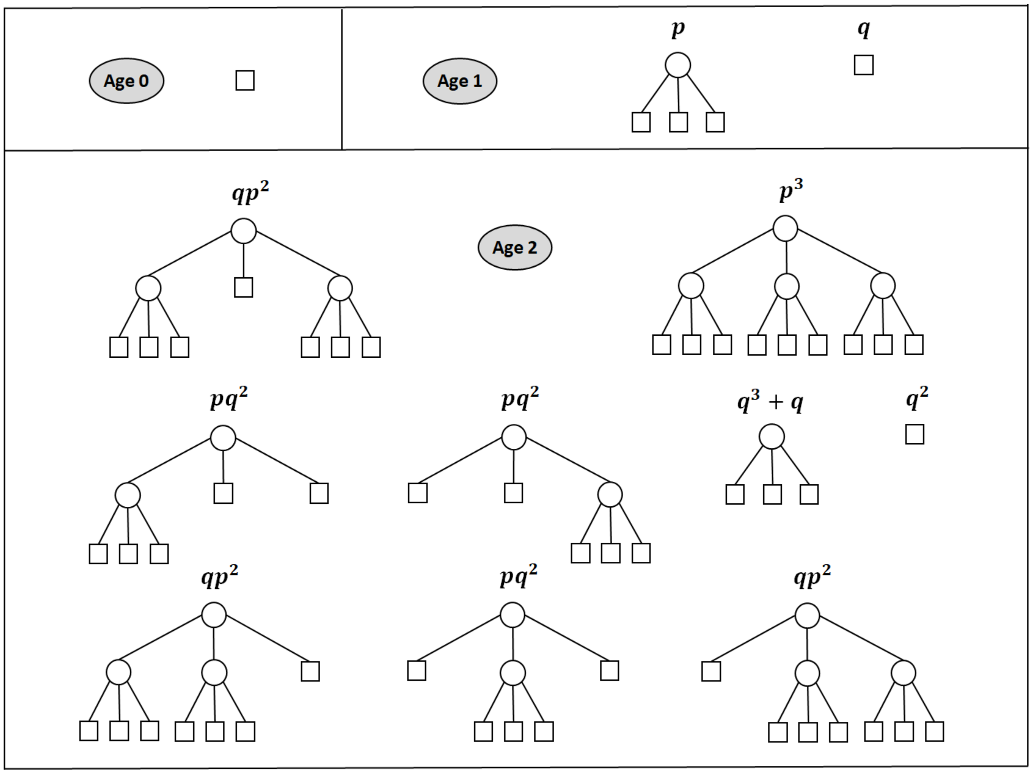

Figure 1: All exponential recursive ternary trees of index p and ages 0, 1 and 2. The probability of getting each tree appears above the root of that tree.

The ERT generalizes the recursive tree by allowing multiple nodes to grow simultaneously.

We use the term ERT following Mahmoud (2022). These trees are *recursive* in the sense that their construction is recursively defined: a tree of age n is built from independent copies of trees of age $n-1$ via a self-similar branching rule. This recursive structure is central to the distributional recurrences we derive in Sections 2–4. While the growth mechanism is parallel rather than sequential, the recursive nature of the tree is preserved.

In Ghasemi et al. (2023), an *exponential recursive k-ary tree* has been defined as an ERT in which every node has at most k children. The nodes that already exist in the tree are called *internal*. In order to identify the locations of possible insertions of a new node, an exponential recursive k-ary tree is extended by having $k-d$ *external* nodes joined as children of a node with d children. Internal and external nodes are shown by circles and squares, respectively (see, e.g., the trees in Figure 1).

An *exponential recursive k-ary tree of index p and age n* grows as follows: Initially (at age 0), there is an external node. For $n \geq 1$, at the nth step (at age n), every external node independently is changed into a leaf with probability $p \in (0,1)$, or not to be changed with probability $q := 1-p$. The parameter p is called the *index* of the ERT. The *age* of an ERT is the number of steps it takes to grow the tree. Figure 1 shows all exponential recursive 3-ary trees of ages 0, 1 and 2.

We assume $0 < p < 1$ throughout, which ensures non-degenerate stochastic growth. The boundary cases $p = 0$ (no growth) and $p = 1$ (deterministic complete tree) are trivial and can be obtained as limiting cases of our results.

The *Wiener index*, a distance-based topological index, is defined as the sum of distances between all nodes in a graph. For recursive trees, Neininger (2002) has obtained the expectations, the variances and a bivariate limit law for the Wiener index and the total path length. It also has been obtained a strong limit theorem for the Wiener index of recursive trees in Grübel and Michailow (2015).

The *external Wiener index*, the sum of distances between all pairs of external nodes, provides a natural measure of the spread or compactness of the tree's boundary. This is particularly relevant for:

- Network analysis: In social networks modeled by exponential recursive trees, external nodes represent potential future connections or unoccupied slots. The external Wiener index captures the expected distance between these potential connection points, which is crucial for understanding information propagation, epidemic spreading, and network resilience.

- Population genetics and phylogenetics: External nodes in these trees correspond to extant species or individuals. The external Wiener index measures the genetic or phylogenetic distance among current population members, providing insights into diversity and evolutionary history.

- Algorithmic applications: In data structures such as digital search trees or trie structures, external nodes represent empty positions. The external Wiener index quantifies search or insertion costs, making it practically relevant for performance analysis.

By *saturated nodes*, we mean the nodes with 3 children in an exponential recursive ternary tree. A saturated node represents a full branching point. These nodes are of significant interest because:

- They indicate complete local structures, which are critical for understanding the tree's capacity and growth bottlenecks.

- In biological systems, saturated nodes may represent speciation events where all possible descendant lineages are present.

- In network models, saturated nodes correspond to fully connected local hubs, which affect robustness and clustering coefficients.

The limiting distribution of depth and the expectation and variance of *external path length* (the sum of depth of all external nodes) in recursive k-ary trees have been obtained in Javanian and Vahidi-Asl (2006), Javanian and Vahidi-Asl (2012) and Javanian (2014), respectively. The size, leaves, external path length and depth of a randomly chosen external node in exponential recursive k-ary tree have been investigated; see Ghasemi et al. (2023). The asymptotic behaviour of *profile* (number of nodes of the same depth) of exponential binary trees has been studied by Feng and Mahmoud (2018).

In this paper, for the sake of simplicity, we consider the exponential recursive k-ary tree for which $k=3$. The resulting tree is called *exponential recursive ternary tree* (ERTT). Our interest here is to study the external Wiener index W_n and number of saturated nodes Y_n in an ERTT of age n. We give the first two moments of W_n and Y_n. Then we show convergence in distribution and in L^2 of $\frac{W_n}{n(2p+1)^{2n}}$ that is a martingale with finite second moment. By the contraction method, we also characterize the limiting distribution of $\frac{Y_n}{(2p+1)^n}$ as the unique fixed point of a distributional equation.

We use the recursive structure of the ERTTs under consideration in order to setup some recurrences for Wiener index and the number of saturated nodes. After one step of the evolution of an ERTT, the initial external node either succeeds to recruit a leaf (event $\mathscr{R}$) or not. We denote the indicator function of the event $\mathscr{R}$ by $\mathbb{I}$. If the event $\mathscr{R}$ does not occur, then after $n-1$ steps, the initial external node of the tree fathers an ERTT $\tilde{T}_{n-1}$. Alternatively, If $\mathscr{R}$ occurs, the extended ERTT has an internal node with 3 external nodes. By time n, these 3 external nodes produce 3 independent ERTTs T'_{n-1}, T''_{n-1} and T'''_{n-1}. Now, we hook T'_{n-1}, T''_{n-1} and T'''_{n-1} to construct an ERTT of index p and age n. Let $(\tilde{X}_{n-1},\tilde{Z}_{n-1})$, (X'_{n-1},Z'_{n-1}), (X''_{n-1},Z''_{n-1}) and (X'''_{n-1},Z'''_{n-1}), be the pairs of the number of external nodes and the external path length in $\tilde{T}_{n-1}$, T'_{n-1}, T''_{n-1} and T'''_{n-1}, respectively. Then for an ERTT of index p at age n with the number of saturated nodes Y_n and the Wiener index W_n, we have

$$X_n \stackrel{d}{=} (1-\mathbb{I})\tilde{X}_{n-1} + \mathbb{I}\Big(X'_{n-1} + X''_{n-1} + X'''_{n-1}\Big), \tag{1}$$

$$Z_n \stackrel{d}{=} (1-\mathbb{I})\tilde{Z}_{n-1} + \mathbb{I}\Big(Z'_{n-1} + Z''_{n-1} + Z'''_{n-1} + X'_{n-1} + X''_{n-1} + X'''_{n-1}\Big), \tag{2}$$

where the symbol $\stackrel{d}{=}$ denotes equality in distribution; the pairs $(\tilde{X}_{n-1}, \tilde{Z}_{n-1})$, (X'_{n-1}, Z'_{n-1}), (X''_{n-1}, Z''_{n-1}) and (X'''_{n-1}, Z'''_{n-1}) are independent copies of (X_{n-1}, Z_{n-1}) and independent of $\mathbb{I}$. Moreover, the almost sure and L^2 convergence of the scaled X_n and Z_n have been shown in Ghasemi et al. (2023),

$$\frac{X_n}{(2p+1)^n} \xrightarrow{\text{a.s.}} X_*, \qquad \frac{Z_n}{n(2p+1)^n} \xrightarrow{L^2 \ \& \ \text{a.s.}} X_*, \qquad \left(\mathbb{E}[X_*] = 1,\ \mathbb{E}[X_*^2] = \frac{3}{2p+1}\right), \tag{3}$$

as $n \to \infty$, for a limiting variable X_* characterized by inductively-constructed moments.

To setup the recurrences for the number of saturated nodes and Wiener index, let T_n be an ERTT of index p at age n, $(\tilde{Y}_{n-1}, \tilde{W}_{n-1})$, (Y'_{n-1}, W'_{n-1}), (Y''_{n-1}, W''_{n-1}) and (Y'''_{n-1}, W'''_{n-1}) be the pairs of the number of saturated nodes and the Wiener index in $\tilde{T}_{n-1}$, T'_{n-1}, T''_{n-1} and T'''_{n-1}, respectively. Let $\mathbb{J}_{n-1}$, $\mathbb{J}'_{n-1}$, $\mathbb{J}''_{n-1}$ and $\mathbb{J}'''_{n-1}$ be the indicators of the events that, after $n-1$ extra steps, the initial external node succeeds to recruit a child in T_{n-1}, T'_{n-1}, T''_{n-1} and T'''_{n-1}, respectively. The term $\mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}$ appears because a new saturated node is created at the root exactly when all three subtrees have grown from their respective initial external nodes. The factor $\mathbb{I}$ indicates whether the initial external node successfully recruited a child. When this happens, the saturated nodes in the resulting tree are the sum of saturated nodes in the three subtrees plus one additional saturated node if all three subtrees are non-empty (i.e., all three $\mathbb{J}$'s equal 1). We will include the derivation:

$$Y_n \stackrel{d}{=} (1-\mathbb{I})\tilde{Y}_{n-1} + \mathbb{I}\Big(Y'_{n-1} + Y''_{n-1} + Y'''_{n-1} + \mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}\Big). \tag{4}$$

Now, we explain the interaction terms such as $X'_{n-1}X''_{n-1}$, $Z'_{n-1}X''_{n-1}$, etc. These arise from distances between external nodes in different subtrees. For example, if an external node in subtree T'_{n-1} and an external node in subtree T''_{n-1} are considered, their distance is:

$$D(i,j) = D_{T'}(i) + 2 + D_{T''}(j).$$

Summing over all pairs gives the cross terms $Z'_{n-1}X''_{n-1} + Z''_{n-1}X'_{n-1} + 2X'_{n-1}X''_{n-1}$. The factor 2 appears in the constant term because the path goes up from the external node in T' to its root (distance $D_{T'}(i)$), then crosses through the common parent node (distance 2), and goes down to the external node in T'' (distance $D_{T''}(j)$). The $Z'X''$ and $Z''X'$ terms are the sums of $D_{T'}(i)$ over all pairs and $D_{T''}(j)$ over all pairs, respectively. Then we obtain

$$\begin{aligned} W_n \stackrel{d}{=} (1-\mathbb{I})\tilde{W}_{n-1} + \mathbb{I}\Big(& W'_{n-1} + W''_{n-1} + W'''_{n-1} + 2X'_{n-1}X''_{n-1} + 2X'_{n-1}X'''_{n-1} + 2X''_{n-1}X'''_{n-1} \\ & + Z'_{n-1}X''_{n-1} + Z''_{n-1}X'_{n-1} + Z'_{n-1}X'''_{n-1} + Z'''_{n-1}X'_{n-1} + Z''_{n-1}X'''_{n-1} + Z'''_{n-1}X''_{n-1}\Big), \end{aligned} \tag{5}$$

where the triples $(Y'_{n-1}, \mathbb{J}'_{n-1}, W'_{n-1})$, $(Y''_{n-1}, \mathbb{J}''_{n-1}, W''_{n-1})$ and $(Y'''_{n-1}, \mathbb{J}'''_{n-1}, W'''_{n-1})$ are independent copies of $(Y_{n-1}, \mathbb{J}_{n-1}, W_{n-1})$ and independent of $\mathbb{I}$.

In Section 2, we calculate the first two moments of Y_n and W_n. If S_n be the size of be an ERTT of index p at age n, in Section 3 and Section 4, we use the martingale convergence theorem and contraction method to show the following two types of convergence for W_n and Y_n, respectively,

$$\frac{\mathbb{E}[Y_n]}{\mathbb{E}[S_n]} \le \frac{1}{3}, \qquad \frac{Y_n}{(2p+1)^n} \xrightarrow{\text{d}} \hat{Y} \ \text{ and } \qquad \frac{W_n}{n(2p+1)^{2n}} \xrightarrow{L^2 \ \& \ \text{a.s.}} \frac{3p}{2p+1}X_*^2,$$

where the random variable $\hat{Y}$ is the unique distributional fixed-point of a distributional recurrence.

Throughout the paper, we use a systematic notation for the three independent subtrees that arise when the root recruits a child: variables with a single prime (e.g., X'_{n-1}), double prime (e.g., X''_{n-1}), and triple prime (e.g., X'''_{n-1}) represent the three independent copies. Variables with a tilde (e.g., $\tilde{X}_{n-1}$) represent the 'no-recruitment' case. This convention is consistent across all parameters: X for external nodes, Z for external path length, W for Wiener index, Y for saturated nodes, and J for growth indicators. Thus, a reader encountering a symbol such as W''_{n-1} can immediately infer that it refers to the Wiener index of the second independent subtree T''_{n-1}.

The present work makes several novel contributions to the study of exponential recursive ternary trees. First, while Ghasemi et al. (2023) established convergence results for the external path length, we investigate the external Wiener index which provides a more complete description of the tree's global geometric structure. The Wiener index is of fundamental interest in chemical graph theory, network analysis, and phylogenetics, and our results give the first asymptotic characterization of this quantity for exponential recursive trees. Second, we introduce and analyze the number of saturated nodes (nodes with exactly 3 children), a parameter not previously studied in this context. Saturated nodes correspond to fully occupied branching points and have natural interpretations in population genetics (complete speciation events) and network science (local hubs). Third, our methodological contributions include: (i) a martingale construction involving second-order terms to prove almost sure and L^2 convergence of the scaled Wiener index; and (ii) an application of the contraction method to establish the limiting distribution of the number of saturated nodes. These results extend and complement those of Ghasemi et al. (2023), which focused on external path length and size, and they provide a foundation for further study of distance-based parameters in exponential recursive trees.

2 The first two moments

All algebraic simplifications in this section are obtained by direct substitution and summation of geometric series. No specialized numerical or algorithmic code is required; the computations follow from the recurrences stated explicitly in equations (1), (2), (4) and (5). We first need some results that the authors of Ghasemi et al. (2023) have found

$$\mathbb{E}[X_n] = (2p+1)^n, \quad \mathbb{E}[X_n^2] = 3(2p+1)^{2n-1} - 2(1-p)(2p+1)^{n-1} \quad \text{and} \quad \mathbb{E}[Z_n] = 3pn(2p+1)^{n-1}.$$

Theorem 1. *Let Y_n be the number of saturated nodes in an ERTT of age n and index p. We have*

$$\mathbb{E}[Y_n] = \frac{(p^3-p^2+p+2)(2p+1)^n}{2(4-p)(p^2-3p+5)} - \frac{1}{2} + (1-p)^n - \frac{3(1-p)^{2n}}{4-p} + \frac{(1-p)^{3n}}{p^2-3p+5}, \qquad n \geq 1,$$

$$\mathbb{E}[Y_n^2] = \frac{3(p^3-p^2+p+2)^2(2p+1)^{2n-1}}{4(4-p)^2(p^2-3p+5)^2} + \mathscr{O}\big((2p+1)^n\big), \qquad \text{as } n \to \infty.$$

Proof. Taking the expectation of the equation (4), one gets

$$\mathbb{E}[Y_n] = \big(2p+1\big)\mathbb{E}[Y_{n-1}] + p\big(1-(1-p)^{n-1}\big)^3,$$

with initial condition $\mathbb{E}[Y_0] = 0$. Iterating this recurrence we find

$$\mathbb{E}[Y_n] = \sum_{i=1}^{n} \big(2p+1\big)^{n-i} p\big(1-(1-p)^{i-1}\big)^3.$$

This sum yields the explicit solution as stated in the theorem.

Raise both sides of the distributional equation (4) to the second power and take expectations:

$$\mathbb{E}[Y_n^2] = \mathbb{E}\Big[(1-\mathbb{I})\tilde{Y}_{n-1}^2 + \mathbb{I}\Big(\big(Y'_{n-1}\big)^2 + \big(Y''_{n-1}\big)^2 + \big(Y'''_{n-1}\big)^2 + \mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1} + 2Y'_{n-1}Y''_{n-1} + 2Y'_{n-1}Y'''_{n-1}$$
$$+ 2Y''_{n-1}Y'''_{n-1} + 2\big(Y'_{n-1}\mathbb{J}'_{n-1}\big)\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1} + 2\big(Y''_{n-1}\mathbb{J}''_{n-1}\big)\mathbb{J}'_{n-1}\mathbb{J}'''_{n-1} + 2\big(Y'''_{n-1}\mathbb{J}'''_{n-1}\big)\mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\Big)\Big],$$

where $(Y'_{n-1}, \mathbb{J}'_{n-1})$, $(Y''_{n-1}, \mathbb{J}''_{n-1})$ and $(Y'''_{n-1}, \mathbb{J}'''_{n-1})$ are independent copies of $(Y_{n-1}, \mathbb{J}_{n-1})$ and independent of $\mathbb{I}$. Then

$$\mathbb{E}[Y_n^2] = (2p+1)\mathbb{E}\big[Y_{n-1}^2\big] + 6p\big(\mathbb{E}[Y_{n-1}]\big)^2 + 6p\big(1-(1-p)^{n-1}\big)^2\mathbb{E}[Y_{n-1}\mathbb{J}_{n-1}] + p\big(1-(1-p)^{n-1}\big)^3.$$

With the explicit substitution of $\mathbb{E}[Y_{n-1}]$ and by $Y_n\mathbb{J}_n = Y_n$, we have

$$\begin{aligned}
\mathbb{E}[Y_n^2] &= \sum_{i=1}^{n}(2p+1)^{n-i}p\Big(6\big(\mathbb{E}[Y_{i-1}]\big)^2 + 6\big(1-(1-p)^{i-1}\big)^2\mathbb{E}[Y_{i-1}] + \big(1-(1-p)^{i-1}\big)^3\Big) \\
&= \frac{3(p^3-p^2+p+2)^2(2p+1)^{2n-1}}{4(4-p)^2(p^2-3p+5)^2} + \frac{6(2+p)((2p+1)^n-(1-p)^{5n})}{(p-4)(p^2-3p+5)(p^4-5p^3+10p^2-10p+7)} \\
&\quad + \frac{3(p^3-p^2+p+2)(p^5-4p^4+5p^3+8p^2-21p+14)(2p+1)^n}{(p-2)(2p+1)(p-4)^2(p^2-3p+5)^2(p^2-3p+3)} - (1-p)^n + \frac{6(1-p)^{2n}}{4-p} \\
&\quad + \frac{(3p^7-18p^6+23p^5-12p^4+49p^3-284p^2+234p+194)(2p+1)^n}{2(2p+1)(p-4)^2(p^2-3p+5)^2} + \frac{1}{4} - \frac{5(1-p)^{3n}}{p^2-3p+5} \\
&\quad + \frac{18(1-p)((2p+1)^n-(1-p)^{4n})}{(p^3-4p^2+6p-6)(p-4)^2} - \frac{6((2p+1)^n-(1-p)^{6n})}{(p^5-6p^4+15p^3-20p^2+15p-8)(p^2-3p+5)^2} \\
&\quad + \frac{3(2+p)(p^3-p^2+p+2)(1-p)^{2n}(2p+1)^{n-1}}{(2-p)(p-4)^2(p^2-3p+3)(p^2-3p+5)} + \frac{6(p^3-p^2+p+2)(1-p)^{3n}(2p+1)^{n-1}}{(p-4)(p^2-3p+3)(p^2-3p+5)^2},
\end{aligned}$$

where the calculation has been done by MAPLE software (see the following Remark). □

Remark 1. *Regarding the MAPLE computations, we clarify the nature of our MAPLE usage. The computations in Theorems 1 and 2 involve:*

- *For Theorem 1: Substituting the explicit expression for $\mathbb{E}[Y_{i-1}]$ into the recurrence:*

$$\mathbb{E}[Y_n^2] = \sum_{i=1}^{n}(2p+1)^{n-i}p\Big(6\big(\mathbb{E}[Y_{i-1}]\big)^2 + 6\big(1-(1-p)^{i-1}\big)^2\mathbb{E}[Y_{i-1}] + \big(1-(1-p)^{i-1}\big)^3\Big).$$

  This involves expanding polynomial expressions in $(2p+1)^i$, $(1-p)^i$, and their products, then summing geometric series.

- *For Theorem 2: Solving the recurrence (6) for $\mathbb{E}[W_n^2]$, which requires substituting the explicit solutions for the auxiliary moments $\mathbb{E}[Z_n^2]$, $\mathbb{E}[W_nX_n]$, $\mathbb{E}[W_nZ_n]$ and $\mathbb{E}[Z_nX_n]$, then simplifying the resulting expression.*

The MAPLE operations are straightforward: we used the `sum()` *command to evaluate finite sums and the* `simplify()` *command to combine like terms. There is no specialized algorithm or custom code to share, the computations consist of typing the sums as written in the paper and allowing MAPLE to perform the symbolic algebra. A representative example would be:*

Since the computations involve only basic symbolic summation and simplification, providing a separate MAPLE code file would not meaningfully enhance reproducibility beyond the explicit recurrences already given in the paper. The recurrences themselves are fully specified, and any computer algebra system can reproduce the results by following the same steps.

Let S_n be the size of be an ERTT of index p at age n, then by the result $\mathbb{E}[S_n] \sim \frac{(2p+1)^n}{2}$ in Ghasemi et al. (2023) and Theorem 1, we have

$$\frac{\mathbb{E}[Y_n]}{\mathbb{E}[S_n]} \leq \frac{1}{3}, \qquad \left(\because \quad \frac{p^3 - p^2 + p + 2}{(4-p)(p^2 - 3p + 5)} \leq \frac{1}{3} \right).$$

Theorem 2. *Let W_n be the wiener index in an ERTT of age n and index p. We have*

$$\mathbb{E}[W_n] = \frac{27p^2 n^2 (2p+1)^{2n-2}}{3np + 2p + 1} + \frac{9pn(2p+1)^{2n-1}}{2(3np+2p+1)} - \frac{3(2p+1)^{2n}}{2(3np+2p+1)} + \frac{3}{2}(2p+1)^{n-1}, \qquad n \geq 1,$$

$$\mathbb{E}[W_n^2] = \frac{324p^2(12p^2+31p+35)}{4(p+1)(4p^2+6p+3)} n^2 (2p+1)^{4n-5} + \mathscr{O}\big(n(2p+1)^{4n}\big), \qquad \textit{as } n \to \infty.$$

Proof. Take expectations both sides of the equation (5). By the identical distribution of W_n', W_n'', W_n''' and W_n; and the independence of $\mathbb{I}$, we get

$$\begin{aligned} \mathbb{E}[W_n] &= (2p+1)\mathbb{E}[W_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}] + 6p\big(\mathbb{E}[X_{n-1}]\big)^2 \\ &= (2p+1)\mathbb{E}[W_{n-1}] + 18p^2 n(p+1)^{2n-3} + 6p(1-p)(p+1)^{2n-3}, \end{aligned}$$

with $\mathbb{E}[W_0] = 0$. By iterating, we have

$$\mathbb{E}[W_n] = \sum_{i=1}^{n} (p+1)^{n-i} \Big(18p^2 i (2p+1)^{2i-3} + 6p(1-p)(2p+1)^{2i-3} \Big),$$

that is obtained as claimed in the theorem.

For obtaining $\mathbb{E}[W_n^2]$, raise both sides of (5) to the second power and take expectations. The second moment recurrence takes the form,

$$\begin{aligned} \mathbb{E}[W_n^2] &= (2p+1)\mathbb{E}[W_{n-1}^2] + 6p\mathbb{E}[Z_{n-1}^2]\mathbb{E}[X_{n-1}^2] + 12p\big(\mathbb{E}[X_{n-1}^2]\big)^2 + 24p\mathbb{E}[Z_{n-1}X_{n-1}]\big(\mathbb{E}[X_{n-1}]\big)^2 \\ &\quad + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}]\mathbb{E}[W_{n-1}] + 12p\mathbb{E}[W_{n-1}X_{n-1}]\mathbb{E}[Z_{n-1}] + 24p\mathbb{E}[W_{n-1}X_{n-1}]\mathbb{E}[X_{n-1}] \\ &\quad + 12p\mathbb{E}[W_{n-1}Z_{n-1}]\mathbb{E}[X_{n-1}] + 24p\mathbb{E}[Z_{n-1}X_{n-1}]\mathbb{E}[X_{n-1}^2] + 24p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2]\mathbb{E}[Z_{n-1}] \\ &\quad + 6p\big(\mathbb{E}[Z_{n-1}X_{n-1}]\big)^2 + 12p\mathbb{E}[W_{n-1}]\big(\mathbb{E}[X_{n-1}]\big)^2 + 6p\big(\mathbb{E}[W_{n-1}]\big)^2 + 24p\mathbb{E}[X_{n-1}^2]\big(\mathbb{E}[X_{n-1}]\big)^2 \\ &\quad + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[Z_{n-1}^2]\big(\mathbb{E}[X_{n-1}]\big)^2 + 6p\mathbb{E}[X_{n-1}^2]\big(\mathbb{E}[Z_{n-1}]\big)^2. \end{aligned} \tag{6}$$

In the recurrence (6), we need the expectations $\mathbb{E}[Z_n^2]$, $\mathbb{E}[W_nZ_n]$, $\mathbb{E}[W_nX_n]$ and $\mathbb{E}[Z_nX_n]$.

From (1), (2) and (5), we have the following recurrences, for $n \geq 1$,

$$\mathbb{E}[Z_nX_n] = (2p+1)\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}] + 3p\mathbb{E}[X_{n-1}^2] + 6p\big(\mathbb{E}[X_{n-1}]\big)^2,$$

$$\mathbb{E}[Z_n^2] = (2p+1)\mathbb{E}[Z_{n-1}^2] + 3p\mathbb{E}[X_{n-1}^2] + 6p\big(\mathbb{E}[Z_{n-1}]\big)^2 + 6p\big(\mathbb{E}[X_{n-1}]\big)^2 + 6p\mathbb{E}[Z_{n-1}X_{n-1}] + 12p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}],$$

$$\begin{aligned} \mathbb{E}[W_nX_n] &= (2p+1)\mathbb{E}[W_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\big(\mathbb{E}[X_{n-1}]\big)^2 + 6p\mathbb{E}[W_{n-1}]\mathbb{E}[X_{n-1}] \\ &\quad + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2] + 6p\big(\mathbb{E}[X_{n-1}]\big)^3 \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}[W_nZ_n] &= (2p+1)\mathbb{E}[W_{n-1}Z_{n-1}] + 18p\mathbb{E}[X_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 3p\mathbb{E}[W_{n-1}X_{n-1}] + 6p\big(\mathbb{E}[X_{n-1}]\big)^3 \\ &\quad + 6p\mathbb{E}[Z_{n-1}^2]\mathbb{E}[X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[W_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[Z_{n-1}X_{n-1}] + 6p\mathbb{E}[X_{n-1}]\big(\mathbb{E}[Z_{n-1}]\big)^2 \\ &\quad + 6p\mathbb{E}[W_{n-1}]\mathbb{E}[X_{n-1}] + 6p\mathbb{E}[Z_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[X_{n-1}]\mathbb{E}[X_{n-1}^2] + 12p\mathbb{E}[Z_{n-1}]\big(\mathbb{E}[X_{n-1}]\big)^2, \end{aligned}$$

with initial conditions $\mathbb{E}[Z_0^2]=0$, $\mathbb{E}[W_0Z_0]=0$, $\mathbb{E}[W_0X_0]=0$ and $\mathbb{E}[Z_0X_0]=0$.

The above recurrence equations are of the form $y_n=(p+1)y_{n-1}+h_n$, with solution

$$y_n=\sum_{i=1}^{n}(p+1)^{n-i}h_i. \tag{7}$$

Therefore, their solutions are obtained as follows

$$\mathbb{E}[Z_nX_n]=9pn(2p+1)^{2n-2}+3(1-p)(2p+1)^{2n-2}-6p(1-p)n(2p+1)^{n-2}-3(1-p)(2p+1)^{n-2},$$

$$\begin{aligned}\mathbb{E}[Z_n^2]&=27p^2n^2(2p+1)^{2n-3}+8p(1-p)n(2p+1)^{2n-3}+\frac{3}{2}(1-p)(5-2p)(2p+1)^{2n-3}\\&\quad-18(1-p)p^2n^2(2p+1)^{n-3}-6p(1-p)(4-p)n(2p+1)^{n-3}-\frac{3}{2}(1-p)(5-2p)(2p+1)^{n-3},\end{aligned}$$

$$\begin{aligned}\mathbb{E}[W_nX_n]&=\frac{9p(p+5)}{p+1}n(2p+1)^{3n-3}-\frac{3(2p^3+22p^2+13p-1)}{2(p+1)^2}(2p+1)^{3n-3}-36p(1-p)n(2p+1)^{2n-3}\\&\quad-\frac{3}{2}(8p^2-16p-1)(2p+1)^{2n-3}-\frac{3(1-p)(4p^2+3p+2)}{2(p+1)^2}(2p+1)^{n-2}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[W_nZ_n]&=\frac{27p^2(p+5)}{p+1}n^2(2p+1)^{3n-4}-\frac{9p(4p^3+38p^2+11p-17)}{2(p+2)^2}n(2p+1)^{3n-4}\\&\quad-\frac{3(1-p)(4p^3+62p^2+17p-20)}{4(p+2)^2}(2p+1)^{3n-4}-108p^2(1-p)n^2(2p+1)^{2n-4}\\&\quad-\frac{9}{2}p(20p^2-52p+23)n(2p+1)^{2n-4}-3(1-p)(10p^2-20p+1)(2p+1)^{2n-4}\\&\quad-\frac{9p(1-p)(4p^2+3p+2)}{2(p+2)^2}n(2p+1)^{n-3}+\frac{3(1-p)(20p^3-8p^2-23p-16)}{4(p+2)^2}(2p+1)^{n-3}.\end{aligned}$$

Again, according to (7), the solution to the recurrence (6) is

$$\begin{aligned}\mathbb{E}[W_n^2]&=\frac{324p^2(12p^2+31p+35)}{4(p+1)(4p^2+6p+3)}n^2(2p+1)^{4n-5}\\&\quad-\frac{27p(176p^6+880p^5+2056p^4+2222p^3+927p^2-66p-111)\,n(2p+1)^{4n-5}}{(p+1)^2(4p^2+6p+3)^2}+\mathscr{O}\big((2p+1)^{4n}\big),\end{aligned}$$

as $n\to\infty$. Hence, we obtain the assertion. □

3 Almost sure convergence for the external Wiener index

In T_n, an ERTT of age n and index p, assume we numbered the external nodes of the tree arbitrarily with the numbers $1,2,\dots,X_n$. Let $D_{i,j}$ be the distance between external nodes i and j, for $i,j\in\{1,2,\dots,X_n\}$. The external Wiener index of T_n is

$$W_n=\sum_{\substack{i=1\\i<j}}^{X_n}\sum_{j=1}^{X_n}D_{i,j}.$$

Theorem 3. *Let W_n be the external Wiener index in an ERTT of age n and index p. As $n\to\infty$, we have almost sure convergence and in L^2,*

$$\frac{W_n}{n(2p+1)^{2n}}\xrightarrow{L^2\ \&\ \text{a.s.}}\frac{3p}{2p+1}X_*^2.$$

Proof. Let $\mathbb{I}_k$ be a sequence of independent Bernoulli(p) random variables, defined on the same probability space as the trees. Consider $\{\mathbb{I}_k\}_{k=1}^{\infty}$ to be independent of the structure of the tree at any age, too. If each of two external nodes i and j or both succeed in attracting a new node, then by direct enumeration, the contribution of the external nodes i and j of T_{n-1} into W_n, is

$$9\mathbb{I}_i\mathbb{I}_j(D_{i,j}+2)D_{i,j}\mathbb{I}_{i<j}+3\mathbb{I}_i\big(1-\mathbb{I}_j\big)(D_{i,j}+1)\mathbb{I}_{i\neq j}+6\mathbb{I}_i-\mathbb{I}_i\mathbb{I}_jD_{i,j}\mathbb{I}_{i<j}-\mathbb{I}_i\big(1-\mathbb{I}_j\big)D_{i,j}$$

for $i,j\in\{1,2,\ldots,X_{n-1}\}$.

Consequently, we write

$$\begin{aligned}W_n=W_{n-1}&-\sum_{\substack{i=1\\ i\neq j}}^{X_{n-1}}\sum_{j=1}^{X_{n-1}}\mathbb{I}_i\big(1-\mathbb{I}_j\big)D_{i,j}-\sum_{\substack{i=1\\ i<j}}^{X_{n-1}}\sum_{j=1}^{X_{n-1}}\mathbb{I}_i\mathbb{I}_jD_{i,j}+6\sum_{i=1}^{X_{n-1}}\mathbb{I}_i+3\sum_{\substack{i=1\\ i\neq j}}^{X_{n-1}}\sum_{j=1}^{X_{n-1}}\mathbb{I}_i\big(1-\mathbb{I}_j\big)(D_{i,j}+1)\\&+9\sum_{\substack{i=1\\ i<j}}^{X_{n-1}}\sum_{j=1}^{X_{n-1}}\mathbb{I}_i\mathbb{I}_j(D_{i,j}+2).\end{aligned}$$

Let $\mathscr{F}_n$ be the sigma field generated by the first n steps of the evolution of T_n. Then

$$\begin{aligned}\mathbb{E}[W_n|\mathscr{F}_{n-1}]&=W_{n-1}+4p(1-p)W_{n-1}+8p^2W_{n-1}+6pX_{n-1}+3p(2p+1)X_{n-1}(X_{n-1}-1)\\&=(2p+1)^2W_{n-1}+3p(2p+1)X_{n-1}^2-3p(2p-1)X_{n-1}.\end{aligned}\tag{8}$$

Let $M_n:=\alpha_nW_n+\beta_nX_n^2+\gamma_nX_n$, for specified factors α_n, β_n and γ_n that transformed M_n a martingale. So, by (8) and a martingalization procedure, we have

$$\begin{aligned}\mathbb{E}[M_n|\mathscr{F}_{n-1}]&=\alpha_n(2p+1)^2W_{n-1}+3\alpha_np(2p+1)X_{n-1}^2-3\alpha_np(2p-1)X_{n-1}\\&\quad+\beta_n\mathbb{E}\big[X_n^2\big|\mathscr{F}_{n-1}\big]+\gamma_n\mathbb{E}\big[X_n\big|\mathscr{F}_{n-1}\big]\\&=\alpha_n(2p+1)^2W_{n-1}+3\alpha_np(2p+1)X_{n-1}^2-3\alpha_np(2p-1)X_{n-1}\\&\quad+\beta_n(2p+1)^2X_{n-1}^2+4\beta_np(1-p)X_{n-1}+\gamma_n(2p+1)X_{n-1}\\&=\alpha_n(2p+1)^2W_{n-1}+\big(3\alpha_np(2p+1)+\beta_n(2p+1)^2\big)X_{n-1}^2\\&\quad+\big(\gamma_n(2p+1)+4\beta_np(1-p)-3\alpha_np(2p-1)\big)X_{n-1}\\&=M_{n-1}=\alpha_{n-1}W_{n-1}+\beta_{n-1}X_{n-1}^2+\gamma_{n-1}X_{n-1}.\end{aligned}$$

This is possible, if we choose

$$\begin{aligned}\alpha_{n-1}&=\alpha_n(2p+1)^2,\\\beta_{n-1}&=3\alpha_np(2p+1)+\beta_n(2p+1)^2,\\\gamma_{n-1}&=\gamma_n(2p+1)+4\beta_np(1-p)-3\alpha_np(2p-1).\end{aligned}$$

From these solvable recurrences, we obtain

$$\begin{aligned}\alpha_n&=\frac{\alpha_0}{(2p+1)^{2n}},\\\beta_n&=\frac{\beta_0}{(2p+1)^{2n}}-\frac{3\alpha_0pn}{(2p+1)^{2n+1}},\\\gamma_n&=\frac{3\alpha_0-4(1-p)\beta_0+2(2p+1)\gamma_0}{2(2p+1)^{n+1}}-\frac{6p(1-p)\alpha_0n}{(2p+1)^{2n+2}}-\frac{3\alpha_0-4(1-p)\beta_0}{2(2p+1)^{2n+1}}.\end{aligned}$$

For simplicity, we take $\alpha_0 = 1$ and $\beta_0 = 0 = \gamma_0$. Thus,

$$M_n = \frac{1}{(2p+1)^{2n}} W_n - \frac{3pn}{(2p+1)^{2n+1}} X_n^2 + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right) X_n,$$

is a martingale with bounded moments

$$\begin{aligned}
\mathbb{E}[M_n] &= \frac{\mathbb{E}[W_n]}{(2p+1)^{2n}} - \frac{3pn\mathbb{E}[X_n^2]}{(2p+1)^{2n+1}} + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right) \mathbb{E}[X_n] \\
&= \frac{27p^2n^2(2p+1)^{-2}}{3np+2p+1} + \frac{9pn(2p+1)^{-1}}{2(3np+2p+1)} - \frac{3}{2(3np+2p+1)} + \frac{3}{2(2p+1)^{n+1}} \\
&\quad - 3pn \left(\frac{3}{(2p+1)^2} - \frac{2(1-p)}{(2p+1)^{n+2}} \right) + \frac{3}{2(2p+1)} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{n+2}} \\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[M_n^2] &= \frac{\mathbb{E}[W_n^2]}{(2p+1)^{4n}} + \frac{9p^2n^2\mathbb{E}[X_n^4]}{(2p+1)^{4n+2}} + \left(\frac{3}{2(2p+1)^{n+1}} - \frac{12p(1-p)n + 3(2p+1)}{2(2p+1)^{2n+2}} \right)^2 \mathbb{E}[X_n^2] \\
&\quad + \left(\frac{3}{(2p+1)^{3n+1}} - \frac{12p(1-p)n + 3(2p+1)}{(2p+1)^{4n+2}} \right) \left(\mathbb{E}[W_nX_n] - \frac{3pn\mathbb{E}[X_n^3]}{2p+1} \right) - \frac{6pn\mathbb{E}[W_nX_n^2]}{(2p+1)^{4n+1}} \\
&= \frac{1}{(2p+1)^{4n}} \mathbb{E}[W_n^2] + \frac{9p^2n^2}{(2p+1)^{4n+2}} \mathbb{E}[X_n^4] - \frac{6pn}{(2p+1)^{4n+1}} \mathbb{E}[W_nX_n^2] + \mathscr{O}(1) \\
&= \frac{1}{(2p+1)^{4n}} \left(\frac{324p^2(12p^2+31p+35)}{4(p+1)(4p^2+6p+3)} n^2 (2p+1)^{4n-5} \right. \\
&\quad \left. - \frac{27p(176p^6+880p^5+2056p^4+2222p^3+927p^2-66p-111)\, n(2p+1)^{4n-5}}{(p+1)^2(4p^2+6p+3)^2} \right) \\
&\quad + \frac{9p^2n^2}{(2p+1)^{4n+2}} \left(\frac{9(12p^2+31p+35)(2p+1)^{4n-3}}{(p+1)(4p^2+6p+3)} \right) \\
&\quad - \frac{6pn}{(2p+1)^{4n+1}} \left(\frac{27p(12p^2+31p+35)n(2p+1)^{4n-4}}{(4p^2+6p+3)(p+1)} \right. \\
&\quad \left. - \frac{9(176p^6+880p^5+2056p^4+2222p^3+927p^2-66p-111)}{2(p+1)^2(4p^2+6p+3)^2} (2p+1)^{4n-4} \right) + \mathscr{O}(1) \\
&= 0 + \mathscr{O}(1) < \infty, \qquad \text{as } n \to \infty,
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{E}[X_n^3] &= \frac{3(p+5)(2p+1)^{3n-2}}{p+1} - 18(1-p)(2p+1)^{2n-2} + \frac{2(1-p)(2-p)(2p+1)^{n-1}}{p+1}, \\
\mathbb{E}[X_n^4] &= \frac{9(12p^2+31p+35)(2p+1)^{4n-3}}{(p+1)(4p^2+6p+3)} - \frac{36(p+5)(1-p)(2p+1)^{3n-3}}{p+1} \\
&\quad - \frac{12(1-p)(7p^2-6p-7)(2p+1)^{2n-3}}{p+1} - \frac{8(1-p)(2p^4-12p^3+7p^2+6p+3)(2p+1)^{n-2}}{(p+1)(4p^2+6p+3)},
\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[W_n X_n^2] = {} & \frac{27p(12p^2+31p+35)n(2p+1)^{4n-4}}{(4p^2+6p+3)(p+1)} \\ & - \frac{9(176p^6+880p^5+2056p^4+2222p^3+927p^2-66p-111)}{2(p+1)^2(4p^2+6p+3)^2}(2p+1)^{4n-4} \\ & + \frac{3(60p^4+300p^3-60p^2-300p)n(2p+1)^{3n-4}}{2(p+1)^2} + \mathscr{O}\big((2p+1)^{3n}\big), \quad \text{as } n \to \infty.\end{aligned}$$

So, by the martingale convergence theorem (Theorem 6.6.9 in Ash (1999)), M_n converges to some M_* almost surely and in L^2. Consequently,

$$\begin{aligned}\frac{M_n}{n} &= \frac{W_n}{n(2p+1)^{2n}} - \frac{3p}{2p+1}\left(\frac{X_n}{(2p+1)^n}\right)^2 + \left(\frac{3}{2(2p+1)} - \frac{12p(1-p)n+3(2p+1)}{2(2p+1)^{n+2}}\right)\frac{X_n}{n(2p+1)^n} \\ &\xrightarrow{L^2 \ \& \ \text{a.s.}} 0.\end{aligned}$$

Hence, by (3), the result follows. □

Corollary 1. *Let D_n^* be the distance between two randomly chosen nodes in a random ERTT of age n and index p. Then we have*

$$\frac{\mathbb{E}[D_n^*]}{n} \longrightarrow \frac{6p}{2p+1}.$$

Proof. In T_n, a random ERTT of age n and index p, we have

$$\mathbb{E}[D_n^* | T_n] = \frac{2\sum\limits_{\substack{i=1 \\ i<j}}^{X_n}\sum\limits_{j=1}^{X_n} D_{i,j}}{X_n(X_n-1)} = \frac{2W_n}{X_n(X_n-1)}.$$

Therefore, by (3) and Theorem 3, we have

$$\frac{\mathbb{E}[D_n^* | T_n]}{n} = \frac{2W_n / \big(n(2p+1)^{2n}\big)}{X_n(X_n-1)/((2p+1)^{2n})} \xrightarrow{\text{a.s.}} \frac{6p}{2p+1}.$$

Since $W_n \le X_n(X_n-1)n$, then $\mathbb{E}[D_n^* | T_n]/n \le 2$. Consequently, by dominated convergence theorem,

$$\lim_{n\to\infty} \frac{\mathbb{E}[D_n^*]}{n} = \lim_{n\to\infty} \mathbb{E}\left[\frac{\mathbb{E}[D_n^* | T_n]}{n}\right] = \mathbb{E}\left[\lim_{n\to\infty} \frac{\mathbb{E}[D_n^* | T_n]}{n}\right].$$

This yields the result. □

4 L^2 convergence for the number of saturated nodes

In this section we apply the contraction method in Roesler and Rueschendorf (2001) or the multivariate contraction method in Neininger (2001) to state the limiting distribution of $\hat{Y}_n := \frac{Y_n}{(2p+1)^n}$.

We first need to set up some notation as follows: For a random variable X, we write $X \sim \lambda$ if the distribution of X is λ, i.e. the law $\mathscr{L}(X)$ of X is λ. The symbol $\|X\|_2 := (\mathbb{E}[|X|^2])^{1/2}$ denotes the usual

L_2-norm of X. The Wasserstein-metric ℓ_2 is defined on the space of probability distributions with finite second moments by

$$\ell_2(X,Y) := \ell_2(\mathscr{L}(X), \mathscr{L}(Y)) := \ell_2(\lambda, \nu) := \inf\{\|X-Y\|_2 : X \sim \lambda, Y \sim \nu\}.$$

By $\mathscr{M}_2$ the space of all probability distributions λ with mean $\dfrac{p^3-p^2+p+2}{2(4-p)(p^2-3p+5)}$ (as in Theorem 1) and finite second moment is denoted. The metric space $(\mathscr{M}_2, \ell_2)$ is complete and convergence in ℓ_2 is equivalent to convergence in distribution plus convergence of the second moments.

By equation (4) we have

$$\frac{Y_n}{(2p+1)^n} \stackrel{d}{=} \frac{\mathbb{I}}{2p+1}\left(\frac{Y'_{n-1}}{(2p+1)^{n-1}} + \frac{Y''_{n-1}}{(2p+1)^{n-1}} + \frac{Y'''_{n-1}}{(2p+1)^{n-1}}\right) + \frac{1-\mathbb{I}}{2p+1}\frac{\tilde{Y}_{n-1}}{(2p+1)^{n-1}} + \mathbb{I}\frac{\mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}}{(2p+1)^n}. \quad (9)$$

Theorem 4. *Let Y_n be the saturated nodes of an ERTT of age n and index p. The normalized saturated nodes $\hat{Y}_n := \frac{Y_n}{(2p+1)^n}$ satisfies the distributional recursion*

$$\hat{Y}_n \stackrel{d}{=} \frac{\mathbb{I}}{2p+1}\left(\hat{Y}'_{n-1} + \hat{Y}''_{n-1} + \hat{Y}'''_{n-1}\right) + \frac{1-\mathbb{I}}{2p+1}\bar{Y}_{n-1} + \mathbb{I}\frac{\mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}}{(2p+1)^n}, \tag{10}$$

where $\mathbb{J}'_{n-1} \stackrel{d}{=} \mathbb{J}''_{n-1} \stackrel{d}{=} \mathbb{J}'''_{n-1} \stackrel{d}{=} \mathrm{Bernoulli}(1-(1-p)^{n-1})$, $\bar{Y}_n$, $(\hat{Y}'_n, \mathbb{J}'_{n-1})$, $(\hat{Y}''_n, \mathbb{J}''_{n-1})$, $(\hat{Y}'''_n, \mathbb{J}'''_{n-1})$ and $\mathbb{I}$ are independent, $\bar{Y}_n \stackrel{d}{=} \hat{Y}'_n \stackrel{d}{=} \hat{Y}''_n \stackrel{d}{=} \hat{Y}'''_n \stackrel{d}{=} \hat{Y}_n$, and $\mathbb{I}$ is a Bernoulli random variable with success probability p, then $\hat{Y}_n \xrightarrow{\mathrm{d}} \hat{Y}$, as $n \to \infty$, where the random variable $\hat{Y}$ is the unique distributional fixed-point of

$$\hat{Y} \stackrel{d}{=} \frac{\mathbb{I}}{2p+1}\left(\hat{Y}' + \hat{Y}'' + \hat{Y}'''\right) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}, \tag{11}$$

with $\hat{Y}'$, $\hat{Y}''$ and $\hat{Y}'''$ are independent copies of $\hat{Y}$ and independent of $\mathbb{I}$.

Proof. The equation (10) is an immediate consequence of the equation (9), where $\bar{Y}_n := \frac{\tilde{Y}_n}{(2p+1)^n}$, $\hat{Y}'_n := \frac{Y'_n}{(2p+1)^n}$, $\hat{Y}''_n := \frac{Y''_n}{(2p+1)^n}$ and $\hat{Y}'''_n := \frac{Y'''_n}{(2p+1)^n}$. Using Chebyshev's inequality and $\mathbb{E}[\mathbb{J}''_{n-1}] = \mathbb{E}[\mathbb{J}'''_{n-1}] = 1-(1-p)^{n-1}$, we have the convergence in probability $\frac{\mathbb{J}'_{n-1}\mathbb{J}''_{n-1}\mathbb{J}'''_{n-1}}{(2p+1)^n} \xrightarrow{\mathrm{p}} 0$. That is,

$$\hat{Y}_n \stackrel{d}{=} \frac{\mathbb{I}}{2p+1}\cdot\left(\hat{Y}'_{n-1} + \hat{Y}''_{n-1} + \hat{Y}'''_{n-1}\right) + \frac{1-\mathbb{I}}{2p+1}\cdot\hat{Y}_{n-1} + o_p(1), \tag{12}$$

where the notation $o_p(1)$ will denote an asymptotically negligible random variable that converges to 0 in probability. For purposes of convergence we can, and hence will, ignore the $o_p(1)$ term.

Consider the well-defined transformation $T : \mathscr{M}_2 \to \mathscr{M}_2$,

$$T(\lambda) := \mathscr{L}\Big(\frac{\mathbb{I}}{2p+1}\left(\hat{Y}'_\lambda + \hat{Y}''_\lambda + \hat{Y}'''_\lambda\right) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\lambda\Big), \tag{13}$$

where $\hat{Y}_\lambda \stackrel{d}{=} \hat{Y}$, $\hat{Y}'_\lambda \stackrel{d}{=} \hat{Y}'$, $\hat{Y}''_\lambda \stackrel{d}{=} \hat{Y}''$ and $\hat{Y}'''_\lambda \stackrel{d}{=} \hat{Y}'''$ have λ as distribution and are independent of $\mathbb{I}$.

At a first step, we have to prove that the transformation T has a unique fixed point with respect to the ℓ_2-metric. Let $\lambda, \nu \in \mathscr{M}_2$ be given. By (13), we have

$$T(\lambda) = \mathscr{L}\Big(\frac{\mathbb{I}}{2p+1}\left(\hat{Y}'_\lambda + \hat{Y}''_\lambda + \hat{Y}'''_\lambda\right) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\lambda\Big),$$
$$T(\nu) = \mathscr{L}\Big(\frac{\mathbb{I}}{2p+1}\left(\hat{Y}'_\nu + \hat{Y}''_\nu + \hat{Y}'''_\nu\right) + \frac{1-\mathbb{I}}{2p+1}\hat{Y}_\nu\Big),$$

and

$$\ell_2^2(T(\lambda),T(\nu)) \le 3\,\frac{\mathbb{E}[\mathbb{I}^2]}{(2p+1)^2}\,\mathbb{E}\big[\big|\hat{Y}_\lambda-\hat{Y}_\nu\big|\big]^2+\frac{\mathbb{E}\big[(1-\mathbb{I})^2\big]}{(2p+1)^2}\,\mathbb{E}\big[\big|\hat{Y}_\lambda-\hat{Y}_\nu\big|\big]^2$$
$$=\frac{1}{2p+1}\,\mathbb{E}\big[\big|\hat{Y}_\lambda-\hat{Y}_\nu\big|\big]^2.$$

Therefore, we have

$$\ell_2^2(T(\lambda),T(\nu)) \le \frac{1}{2p+1}\,\ell_2^2(\lambda,\nu).$$

Since $\frac{1}{2p+1}<1$, we deduce that T is a contraction with respect to the ℓ_2-metric. Thus, Banach's fixed point theorem provides existence and uniqueness of solutions of the fixed point equation $T(\lambda)=\lambda$. By (12), if $\hat{Y}_n$ converges in distribution to some random variable $\hat{Y}$, then $\hat{Y}$ satisfies (11). Consequently, the distribution of $\hat{Y}$ will be λ_0, the unique fixed point of T, i.e. $\mathscr{L}(\hat{Y})=T(\lambda_0)=\lambda_0$. Therefore, we have to prove $\lim_{n\to\infty}\ell_2(\hat{Y}_n,\hat{Y})=0$ to conclude $\hat{Y}_n \xrightarrow{\mathrm{d}} \hat{Y}$.

Since $\hat{Y}_n' \stackrel{d}{=} \hat{Y}_n'' \stackrel{d}{=} \hat{Y}_n''' \stackrel{d}{=} \hat{Y}_n$ and $\hat{Y}' \stackrel{d}{=} \hat{Y}'' \stackrel{d}{=} \hat{Y}''' \stackrel{d}{=} \hat{Y}$, we deduce

$$\lim_{n\to\infty}\ell_2^2(\hat{Y}_n,\hat{Y}) \le \frac{p}{(2p+1)^2}\lim_{n\to\infty}\left(\left\|\hat{Y}_{n-1}'-\hat{Y}'\right\|_2^2+\left\|\hat{Y}_{n-1}''-\hat{Y}''\right\|_2^2+\left\|\hat{Y}_{n-1}'''-\hat{Y}'''\right\|_2^2\right)+\frac{1-p}{(2p+1)^2}\lim_{n\to\infty}\left\|\hat{Y}_{n-1}-\hat{Y}\right\|_2^2$$
$$\le \frac{1}{2p+1}\lim_{n\to\infty}\big\|\hat{Y}_n-\hat{Y}\big\|_2.$$

Therefore, we have

$$\lim_{n\to\infty}\ell_2^2(\hat{Y}_n,\hat{Y}) \le \frac{1}{2p+1}\lim_{n\to\infty}\ell_2^2(\hat{Y}_n,\hat{Y}).$$

This is true only if $\lim_{n\to\infty}\ell_2(\hat{Y}_n,\hat{Y})=0$. □

5 Conclusion

This work contributes to the growing body of literature on exponential recursive trees by providing a rigorous probabilistic analysis of two important parameters: the external Wiener index and the number of saturated nodes. The martingale approach and the contraction method, applied in concert, offer a robust framework for studying convergence phenomena in random recursive structures. While we have established convergence in distribution and L^2 for the scaled quantities, the explicit distribution of the limiting random variables X_* and $\hat{Y}$ remains unknown. Characterizing these distributions, perhaps through their moment generating functions or through recursive integral equations, would provide deeper insight into the asymptotic structure of ERTTs.

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