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Credible set estimation for the mean and variance of a normal distribution

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Abstract. In this paper, Bayesian credible sets (or regions) are constructed for the mean and variance of a normal distribution that have smaller areas than the classical exact and approximate regions. These regions are applicable to quality control and other fields, particularly for the joint monitoring of the mean and variance of normal processes. A balanced credible set is derived using conditional posterior distributions. A three-step algorithm is presented to generate posterior samples and determine the highest posterior density (HPD) credible set. The methodology is illustrated and compared with existing approaches using data on the compressive strength of a new alloy.

Keywords: Bayesian credible set; Bayesian estimation; Confidence region; Normal distribution; Prior and posterior distributions.

Mathematics Subject Classification (2020): 62F10, 62F15, 62F25, 62E20.

1 Introduction

Construction of joint confidence regions for normal distribution parameters is very important. These regions provide a range of acceptable values for the mean and variance of the normal distribution based on the observed data at the required confidence level. Therefore, they can be used to jointly monitor the mean and variance of processes that follow a normal distribution.

Joint monitoring of the mean and variance of a normal process has many applications in quality control and applied fields. In quality control, to ensure that a manufacturing process is under control, both the mean and the variance must be within acceptable (and predetermined)

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limits. For example, in the production of car parts such as brake discs, the mean and variance of the disc thickness are monitored against established standards. The mean thickness should fall within a certain range, and the variance should be small (high variance may indicate fluctuations in the production process that should be investigated). In concrete production, monitoring the mean and variance helps to ensure that the mean compressive strength of concrete always remains within the desired range and that fluctuations in concrete quality are reduced. In clinical trials investigating the effectiveness of drugs in treating patients, joint monitoring of the mean and variance can help to evaluate not only the average duration of treatment but also the degree of variability in patients' responses. In the study of income distribution in society, in addition to the average household income, attention must be paid to the variance of income as an indicator of class differences. In all the cases mentioned above, the joint confidence region for the mean and variance is important because it estimates these two important parameters simultaneously and improves our ability to analyze and make decisions.

Suppose that $Y_1, Y_2, \cdots, Y_n$ is a random sample from a normal distribution with mean μ and variance ν, denoted by $\mathcal{N}(\mu, \nu)$, with the probability density function (pdf)

$$f(y;\mu,\nu) = \frac{1}{\sqrt{2\pi\nu}} e^{-\frac{1}{2\nu}(y-\mu)^2}; \quad y \in \mathbb{R},\ \mu \in \mathbb{R},\ \nu > 0,$$

where both μ and ν are unknown. Early studies on joint confidence regions for μ and ν date back to the 1950s. (Mood, 1950, p. 227) obtained an exact confidence region for (μ, ν) using the pivotal method by providing two suitable pivotal quantities. Other available confidence regions for (μ, ν) are approximate (or large-sample) confidence regions, which are mainly obtained using the asymptotic normality of the maximum likelihood estimators (MLEs) of μ and ν. The approximate confidence regions for (μ, ν) and their comparison with Mood's exact region are discussed in Arnold and Shavelle (1998). For confidence regions in other distributions, see, for example, Asgharzadeh et al. (2017, 2020), Fernández (2013, 2014, 2023), Kinaci et al. (2019), Lennartz et al. (2021), Saadati Nik and Asgharzadeh (2025), Saadati Nik et al. (2025) and Wu (2022).

In this paper, we consider Bayesian credible sets for (μ, ν) based on a random sample from the $\mathcal{N}(\mu, \nu)$ distribution. We obtain a balanced credible set for (μ, ν). The HPD credible set for (μ, ν) is also investigated. The proposed Bayesian regions are smaller than the classical exact and approximate confidence regions from the literature.

The organization of the paper is as follows. In Section 2, we review the existing confidence regions for (μ, ν), including both exact and approximate confidence regions. Section 3 discusses the joint prior and posterior distributions for (μ, ν), along with the conditional posterior distributions. In this section, we also derive the balanced and HPD regions for (μ, ν). In Section 4, we provide the areas of the Bayesian regions and compare them with the areas of the exact and large-sample approximate regions. Section 5 provides a numerical example for illustration and comparison.

2 Former confidence regions

Suppose $Y_1, Y_2, \cdots, Y_n$ are a random sample from the $\mathcal{N}(\mu, \nu)$ distribution. Here, we review the exact and approximate confidence regions from the literature. For further details, see Arnold and Shavelle (1998).

2.1 Mood exact region

Mood (1950) derived an exact confidence region for (μ, ν) using the pivotal quantities

$$\frac{\bar{Y} - \mu}{\sqrt{\nu/n}} \sim \mathcal{N}(0,1)$$

and

$$\frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\nu} \sim \chi^2(n-1),$$

where $\bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}$. By choosing numbers a, b, and c such that

$$\Pr\big[-a < \frac{\bar{Y} - \mu}{\sqrt{\nu/n}} < a\big] = \sqrt{1-\zeta},$$

and

$$\Pr\big[b < \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\nu} < c\big] = \sqrt{1-\zeta}.$$

Since $\bar{Y}$ and $\sum_{i=1}^n (Y_i - \bar{Y})^2$ are independent, we obtain

$$\begin{aligned}
&\Pr\Big[-a < \frac{\bar{Y} - \mu}{\sqrt{\nu/n}} < a,\ b < \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\nu} < c\Big] \\
&= \Pr\Big[\bar{Y} - a\sqrt{\frac{\nu}{n}} < \mu < \bar{Y} + a\sqrt{\frac{\nu}{n}},\ \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{c} < \nu < \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{b}\Big] \\
&= \sqrt{1-\zeta}\ \sqrt{1-\zeta} \\
&= 1 - \zeta.
\end{aligned}$$

Specifically, we can consider $a = z_{\frac{1+\sqrt{1-\zeta}}{2}}$, $b = \chi^2_{\frac{1-\sqrt{1-\zeta}}{2}}(n-1)$ and $c = \chi^2_{\frac{1+\sqrt{1-\zeta}}{2}}(n-1)$, where z_γ and $\chi^2_\gamma(n-1)$ are the γ-quantile of the $\mathcal{N}(0,1)$ and $\chi^2(n-1)$ distributions, respectively. This choice gives equal-tail probabilities for both pivot quantities. Thus, a $100(1-\zeta)\%$ exact confidence region for (μ, ν) is the region

$$\mathscr{R}^{\text{Mood}} = \left\{(\mu,\nu):\ \bar{Y} - z_{\zeta_+}\sqrt{\frac{\nu}{n}} < \mu < \bar{Y} + z_{\zeta_+}\sqrt{\frac{\nu}{n}},\ \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\chi^2_{\zeta_+}(n-1)} < \nu < \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{\chi^2_{\zeta_-}(n-1)}\right\}, \quad (1)$$

where, $\zeta_- = \frac{1-\sqrt{1-\zeta}}{2}$ and $\zeta_+ = \frac{1+\sqrt{1-\zeta}}{2}$.

2.2 Approximate confidence region

Based on a random sample $\mathbf{Y}=(Y_1,\ldots,Y_n)$ from the $\mathscr{N}(\mu,\nu)$ distribution, the MLEs of μ and ν, denoted by $\widehat{\mu}$ and $\widehat{\nu}$, are $\widehat{\mu}=\bar{Y}$ and $\widehat{\nu}=S^2/n$, where $S^2=\sum_{i=1}^n(Y_i-\bar{Y})^2$. The approximate confidence region for μ and ν can be obtained using the asymptotic normality of the MLEs $\widehat{\mu}$ and $\widehat{\nu}$.

Under suitable regularity conditions, it is well known that $\sqrt{n}(\widehat{\mu}-\mu,\widehat{\nu}-\nu)'$ is asymptotically a bivariate normal with mean $(0,0)'$ and covariance matrix $I^{-1}(\mu,\nu)$, where

$$I(\mu,\nu)=-\begin{bmatrix} E\Big(\dfrac{\partial^2\ln L(\mu,\nu)}{\partial\mu^2}\Big) & E\Big(\dfrac{\partial^2\ln L(\mu,\nu)}{\partial\mu\partial\nu}\Big)\\ E\Big(\dfrac{\partial^2\ln L(\mu,\nu)}{\partial\nu\partial\mu}\Big) & E\Big(\dfrac{\partial^2\ln L(\mu,\nu)}{\partial\nu^2}\Big)\end{bmatrix}=\begin{pmatrix}\frac{n}{\nu} & 0\\ 0 & \frac{n}{2\nu^2}\end{pmatrix},$$

where $\ln L(\mu,\nu)=\ln f(\mathbf{Y};\mu,\nu)$ is the log-likelihood function. Now, for large n, since the associated quadratic form

$$\mathscr{A}(\mu,\nu)=[\widehat{\mu}-\mu,\widehat{\nu}-\nu]I(\widehat{\mu},\widehat{\nu})[\widehat{\mu}-\mu,\widehat{\nu}-\nu]'$$

has an approximate chi-square distributed with 2 degrees of freedom, it follows that

$$\mathscr{A}(\mu,\nu)=\frac{n}{\widehat{\nu}}(\widehat{\mu}-\mu)^2+\frac{n}{2\widehat{\nu}^2}(\widehat{\nu}-\nu)^2\sim\chi^2(2).$$

Therefore, for large enough n, the region $\Big\{(\mu,\nu):\ \mathscr{A}(\mu,\nu)\le\chi^2_\zeta(2)\Big\}$, or

$$\mathscr{R}^{\text{Approx}}=\Big\{(\mu,\nu):\ \frac{n}{\widehat{\nu}}(\widehat{\mu}-\mu)^2+\frac{n}{2\widehat{\nu}^2}(\widehat{\nu}-\nu)^2\le\chi^2_\zeta(2)\Big\},$$

is a $100(1-\zeta)\%$ approximate confidence region for (μ,ν).

3 Bayesian regions

In this section, Bayesian credible sets for (μ,ν) are obtained.

3.1 Posterior distributions

Suppose that $\mathbf{Y}=(Y_1,\ldots,Y_n)$ is a random sample from the $\mathscr{N}(\mu,\nu)$ distribution with the observed sample $\mathbf{y}=(y_1,\ldots,y_n)$. The likelihood function of $\mathbf{Y}=(Y_1,\ldots,Y_n)$ is

$$\begin{aligned}f(\mathbf{y}\mid\mu,\nu)&=\prod_{i=1}^n f(y_i;\mu,\nu)\\ &\propto\nu^{-\frac{n}{2}}e^{-\frac{1}{2\nu}\sum_{i=1}^n(y_i-\mu)^2}=\nu^{-\frac{n}{2}}e^{-\frac{1}{2\nu}\left[n(\bar{y}-\mu)^2+s^2\right]},\quad y\in\mathbb{R},\ \mu\in\mathbb{R},\ \nu>0,\end{aligned}$$

where $(\bar{y},s^2)$ is the observed value of $(\bar{Y},S^2)$. Clearly $(\bar{Y},S^2)$ is a complete sufficient statistic for (μ,ν). The likelihood function can be written

$$f(\mathbf{y}\mid\mu,\nu)\propto\nu^{-\frac{1}{2}}e^{-\frac{1}{2\nu}[n(\bar{y}-\mu)^2]}\,\nu^{-\frac{n-1}{2}}e^{-\frac{1}{2\nu}s^2}.$$

From (Bolstad, 2010, p. 83), for the joint prior distribution for (μ,ν), let us consider the conditional distribution of μ given ν, and the marginal distribution of ν as

$$h(\mu \mid \nu) \propto \left(\frac{n_0}{\nu}\right)^{\frac{1}{2}} e^{-\frac{n_0}{2\nu}(\mu-m)^2}, \quad \mu \in \mathbb{R};\ m \in \mathbb{R},\ \nu, n_0 > 0,$$
$$h(\nu) \propto \nu^{-\left(\frac{k}{2}+1\right)} e^{-\frac{d}{2\nu}}, \quad \nu > 0;\ k,d > 0,$$

where m, n_0, k and d are the prior parameters (or hyper-parameters). In fact, $h(\mu \mid \nu)$ is the pdf of a $\mathcal{N}(m, \nu/n_0)$ distribution and $h(\nu)$ is the pdf of an inverted gamma distribution with parameters $k/2$ and $d/2$, denoted by InGamma$(k/2, d/2)$. From $h(\mu \mid \nu)$ and $h(\nu)$, the joint conjugate prior pdf for (μ,ν) is obtained as

$$\begin{aligned} h(\mu,\nu) &= h(\mu \mid \nu)\, h(\nu) \\ &\propto \nu^{-(\frac{k+1}{2}+1)} e^{-\frac{1}{2\nu}\left(n_0(\mu-m)^2+d\right)}, \quad \mu \in \mathbb{R},\ \nu > 0. \end{aligned} \tag{2}$$

Note that the kernel of (2) converges to $\nu^{-3/2}$ as $n_0, k, d \to 0$. Thus, in this limiting case, we obtain the following noninformative (improper) joint prior for (μ,ν):

$$h(\mu,\nu) \propto \nu^{-3/2}, \quad \mu \in \mathbb{R},\ \nu > 0.$$

Now, the joint posterior pdf of (μ,ν) given $\mathbf{Y} = \mathbf{y}$ is obtained as

$$\begin{aligned} h(\mu,\nu \mid \mathbf{y}) &\propto f(\mathbf{y} \mid \mu,\nu)\, h(\mu,\nu) \\ &= \nu^{-(\frac{n+k+1}{2}+1)} e^{-\frac{1}{2\nu}\left(n_0(\mu-m)^2+n(\bar{y}-\mu)^2+s^2+d\right)}, \quad \mu \in \mathbb{R},\ \nu > 0, \end{aligned}$$

which can be written as

$$h(\mu,\nu \mid \mathbf{y}) = \frac{1}{c(\mathbf{y})}\, \nu^{-(\frac{k'+1}{2}+1)} e^{-\frac{1}{2\nu}\left(n'(\mu-m')^2+d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\right)}, \quad \mu \in \mathbb{R},\ \nu > 0, \tag{3}$$

where, $k' = n+k$, $d' = d+s^2$, $m' = \frac{n\bar{y}+n_0 m}{n_0+n}$, $n' = n_0+n$ and

$$c(\mathbf{y}) = \int_0^\infty \int_{-\infty}^\infty h(\mu,\nu \mid \mathbf{y})\, \mathrm{d}\mu\, \mathrm{d}\nu = \frac{\sqrt{2\pi}\,\Gamma(\frac{k'}{2})\, 2^{\frac{k'}{2}}}{n'^{\frac{1}{2}}\left(d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\right)^{\frac{k'}{2}}}.$$

For the limiting cases $n_0 \to 0$ and $k, d \to 0$, although the joint prior distribution is improper, but the joint posterior pdf $h(\mu,\nu \mid \mathbf{y})$ is still a proper density.

We now obtain the conditional posterior distribution of ν given $\mathbf{y}$, and conditional posterior distribution of μ given ν and $\mathbf{y}$. These conditional posterior distributions then enable us to generate random samples from the joint posterior distribution $h(\mu,\nu \mid \mathbf{y})$ and obtain Bayesian credible sets.

Corollary 1. *The conditional posterior pdf of* ν *given* $\mathbf{y}$ *is obtained as*

$$h(\nu \mid \mathbf{y}) = \int_{-\infty}^\infty h(\mu,\nu \mid \mathbf{y})\, \mathrm{d}\mu = \frac{\left(d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\right)^{\frac{k'}{2}}}{2^{\frac{k'}{2}}\,\Gamma(\frac{k'}{2})}\, \nu^{-\frac{k'}{2}-1} e^{-\frac{1}{2\nu}\left(d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\right)}, \quad \nu > 0.$$

Clearly, $\nu \mid \mathbf{y} \sim$ *InGamma*$(k'/2,\ b/2)$, *with* $b = d' + \frac{n_0 n}{n_0+n}(\bar{y}-m)^2$.

Corollary 2. *From the posterior distribution (3), the conditional posterior pdf of* μ *given* ν *and* $\mathbf{y}$ *can be found as*

$$h(\mu \mid \nu, \mathbf{y}) = \frac{h(\mu, \nu \mid \mathbf{y})}{h(\nu \mid \mathbf{y})} = \frac{n'^{\frac{1}{2}}}{\sqrt{2\pi\nu}} e^{-\frac{n'(\mu-m')^2}{2\nu}}, \quad \mu \in \mathbb{R}.$$

Clearly, $\mu \mid (\nu, \mathbf{y}) \sim \mathcal{N}(m', \nu/n')$.

3.2 Balanced Bayesian region

Here, a $100(1-\zeta)\%$ balanced Bayesian region for (μ, ν) is proposed using the following two-step algorithm.

Step 1. Select an equal-tailed $100(\sqrt{1-\zeta})\%$ credible interval $\mathscr{B}$ for ν based on the marginal posterior $h(\nu \mid \mathbf{y})$ in Corollary 1. Since $\nu \mid \mathbf{y} \sim \text{InGamma}(k'/2,\ b/2)$, it follows that for given $\mathbf{y}$, $b/\nu \sim \chi^2(k')$ and as a result we have

$$\Pr\left[\chi^2_{\zeta_-}(k') < \frac{b}{\nu} < \chi^2_{\zeta_+}(k')\right] = \sqrt{1-\zeta},$$

or

$$\Pr\left[\frac{d' + \frac{n_0 n}{n_0+n}(\bar{y}-m)^2}{\chi^2_{\zeta_+}(k')} < \nu < \frac{d' + \frac{n_0 n}{n_0+n}(\bar{y}-m)^2}{\chi^2_{\zeta_-}(k')}\right] = \sqrt{1-\zeta}.$$

Therefore, we can propose $\mathscr{B}$ as

$$\mathscr{B} = \left[\frac{d' + \frac{n_0 n}{n_0+n}(\bar{y}-m)^2}{\chi^2_{\zeta_+}(k')},\ \frac{d' + \frac{n_0 n}{n_0+n}(\bar{y}-m)^2}{\chi^2_{\zeta_-}(k')}\right].$$

Step 2. For each $\nu \in \mathscr{B}$, find an equal-tailed $100(\sqrt{1-\zeta})\%$ credible interval $\mathscr{D}(\nu)$ for μ based on the conditional distribution μ given ν and $\mathbf{y}$ in Corollary 2. Since $\mu \mid (\nu, \mathbf{y}) \sim \mathcal{N}(m', \nu/n')$, it therefore follows that under the posterior distribution

$$\frac{\mu - m'}{\sqrt{\frac{\nu}{n'}}} \sim \mathcal{N}(0,1),$$

and we can obtain $\mathscr{D}(\nu)$ as

$$\mathscr{D}(\nu) = \left\{\nu \in \mathscr{B} :\ m' - z_{\zeta_+}\sqrt{\frac{\nu}{n'}} < \mu < m' + z_{\zeta_+}\sqrt{\frac{\nu}{n'}}\right\}.$$

Finally, a $100(1-\zeta)\%$ balanced Bayesian region for (μ, ν) is obtained as

$$\mathscr{R}^{\text{Bayes}} = \left\{(\nu, \mu) :\ \nu \in \mathscr{B},\ m' - z_{\zeta_+}\sqrt{\frac{\nu}{n'}} < \mu < m' + z_{\zeta_+}\sqrt{\frac{\nu}{n'}}\right\}, \tag{4}$$

since

$$\begin{aligned}\Pr\big((\mu,\nu)\in\mathscr{R}^{\text{Bayes}}\mid\mathbf{y}\big) &= \int_{\mathscr{B}} h(\nu\mid\mathbf{y})\Big\{\int_{\mathscr{D}(\nu)} h(\mu\mid\nu,\mathbf{y})\,\mathrm{d}\mu\Big\}\,\mathrm{d}\nu\\ &= \int_{\mathscr{B}} h(\nu\mid\mathbf{y})\,(\sqrt{1-\zeta})\,\mathrm{d}\nu\\ &= (\sqrt{1-\zeta})(\sqrt{1-\zeta}) = 1-\zeta.\end{aligned}$$

Remark 1. *For the limiting case $n_0\to 0$ and $k,d\to 0$, the above $100(1-\zeta)\%$ balanced Bayesian region reduces to*

$$\mathscr{R}^{Bayes} = \Big\{(\mu,\nu):\ \bar{y}-z_{\zeta_+}\sqrt{\frac{\nu}{n}} < \mu < \bar{y}+z_{\zeta_+}\sqrt{\frac{\nu}{n}},\quad \frac{s^2}{\chi^2_{\zeta_+}(n)} < \nu < \frac{s^2}{\chi^2_{\zeta_-}(n)}\Big\},$$

which closely resembles the classical Mood region $\mathscr{R}^{Mood}$ given in (1).

3.3 Bayesian HPD region

Bayesian HPD regions contain the highest-density points of the posterior distribution and are typically the smallest credible sets (see, for example, Casella and Berger 2002; Robert 2001). Here we obtain a Bayesian HPD region for (μ,ν) using the posterior density $h(\mu,\nu\mid\mathbf{y})$.

The Bayesian HPD region for (μ,ν) with a credibility of $(1-\zeta)$, where $\zeta\in(0,1)$, is

$$\begin{aligned}\mathscr{R}^{\text{HPD}} &= \big\{(\mu,\nu)\in\mathbb{R}\times\mathbb{R}^+:\ h(\mu,\nu\mid\mathbf{y})\ge q_{1-\zeta}\big\}\\ &= \Big\{(\mu,\nu)\in\mathbb{R}\times\mathbb{R}^+:\ \frac{1}{c(\mathbf{y})}\,\nu^{-(\frac{k'+1}{2}+1)}\,e^{-\frac{1}{2\nu}\left(n'(\mu-m')^2+d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\right)}\ge q_{1-\zeta}\Big\},\end{aligned}$$

such that

$$\Pr\big((\mu,\nu)\in\mathscr{R}^{\text{HPD}}\mid\mathbf{x}\big) = \iint_{\mathscr{R}^{\text{HPD}}} h(\mu,\nu\mid\mathbf{y})\,\mathrm{d}\mu\,\mathrm{d}\nu = 1-\zeta,$$

where $q_{1-\zeta}$ denotes the ζ-quantile of the random posterior pdf $h(\mu,\nu\mid\mathbf{y})$.

Remark 2. *For the limiting case $n_0\to 0$ and $k,d\to 0$, the HPD region $\mathscr{R}^{HPD}$ reduces to*

$$\mathscr{R}^{HPD} = \Big\{(\mu,\nu)\in\mathbb{R}\times\mathbb{R}^+:\ c'(\mathbf{y})\,\nu^{-(\frac{n+1}{2}+1)}e^{-\frac{1}{2\nu}\left(n(\mu-\bar{y})^2+s^2\right)}\ge q_{1-\zeta}\Big\},$$

where $c'(\mathbf{y}) = \frac{\sqrt{n}s^n}{\sqrt{2\pi}\Gamma(\frac{n}{2})2^{\frac{n}{2}}}$.

Since the exact value of $q_{1-\zeta}$ is not easy to obtain, an approximate of $q_{1-\zeta}$ can be computed through simulation; see, e.g., Fernández (2023). Since $q_{1-\zeta}$ denotes the ζ-quantile of the random variable $h(\mu,\nu\mid\mathbf{y})$, we can generate a sample of random pairs (μ_j,ν_j) for $j=1,2,...,\mathscr{M}$ (where $\mathscr{M}$ is large) from the posterior density $h(\mu,\nu\mid\mathbf{y})$. The ζ-quantile of the simulated sample $\{h(\mu_j,\nu_j\mid\mathbf{y})\}$ for $j=1,2,...,\mathscr{M}$ can be then used as an approximation for $q_{1-\zeta}$.

Note that for the given prior parameters m, n_0, k and d and the available data $\mathbf{y}=(y_1,\ldots,y_n)$, we can use the following three-step algorithm to generate the random sample (μ_j,ν_j) for $j=1,2,...,\mathscr{M}$ from the posterior distribution $h(\mu,\nu\mid\mathbf{y})$:

Algorithm 1

Step 1: Based on Corollary 1, generate ν_1 from the InGamma$(k'/2,\ b/2)$ distribution.
Step 2: Using Corollary 2, generate an observation μ_1 from the $\mathcal{N}(m', \nu_1/n')$ distribution.
Step 3: Repeat Steps 1 and 2, $\mathscr{M}$ times to generate a simulated random sample

$$(\mu_1,\nu_1),\ (\mu_2,\nu_2),\ldots,(\mu_{\mathscr{M}},\nu_{\mathscr{M}})$$

from the posterior density function $h(\mu,\nu \mid \mathbf{y})$.

4 Area of the regions

In this section, we calculate the areas of the $\%100(1-\zeta)$ Bayesian regions for (μ,ν) and compare them with Mood's exact and approximate regions.

The area of the HPD region is

$$\text{Area}(\mathscr{R}^{\text{HPD}}) = \iint\limits_{\mathscr{R}^{\text{HPD}}} \mathrm{d}\mu\,\mathrm{d}\nu.$$

The area of the $100(1-\zeta)\%$ balanced Bayesian region for (μ,ν) in (4) is

$$\begin{aligned}\text{Area}(\mathscr{R}^{\text{Bayes}}) &= \int_{\mathscr{B}}\int_{m'-z_{\zeta_+}\sqrt{\frac{\nu}{n'}}}^{m'+z_{\zeta_+}\sqrt{\frac{\nu}{n'}}} \mathrm{d}\mu\,\mathrm{d}\nu \\ &= \frac{2z_{\zeta_+}}{\sqrt{n'}}\int_{\mathscr{B}}\sqrt{\nu}\,d\nu \\ &= \frac{4z_{\zeta_+}}{3\sqrt{n'}}\big(d'+\frac{n_0 n}{n_0+n}(\bar{y}-m)^2\big)^{3/2}\Big[\big(\frac{1}{\chi^2_{\zeta_-}(k')}\big)^{3/2}-\big(\frac{1}{\chi^2_{\zeta_+}(k')}\big)^{3/2}\Big],\end{aligned}$$

which depends on the sample mean $\bar{y}$ and the sample variance s^2 (since $d' = d+s^2$). For the limiting case where $n_0 \to 0$ and $k,d \to 0$, the area of $\mathscr{R}^{\text{Bayes}}$ becomes:

$$\text{Area}(\mathscr{R}^{\text{Bayes}}) = \frac{4z_{\zeta_+}}{3\sqrt{n}}\left[\left(\frac{1}{\chi^2_{\zeta_-}(n)}\right)^{3/2}-\left(\frac{1}{\chi^2_{\zeta_+}(n)}\right)^{3/2}\right]s^3,$$

which is independent of $\bar{y}$ and is a constant multiple of s^3.

On the other hand, from Arnold and Shavelle (1998), the area of the Mood region is

$$\text{Area}(\mathscr{R}^{\text{Mood}}) = \iint\limits_{\mathscr{R}^{\text{Mood}}} \mathrm{d}\mu\,\mathrm{d}\nu = \frac{4z_{\zeta_+}}{3\sqrt{n}}\Big[\big(\frac{1}{\chi^2_{\zeta_-}(n-1)}\big)^{3/2}-\big(\frac{1}{\chi^2_{\zeta_+}(n-1)}\big)^{3/2}\Big]s^3,$$

and the area of the approximate region is

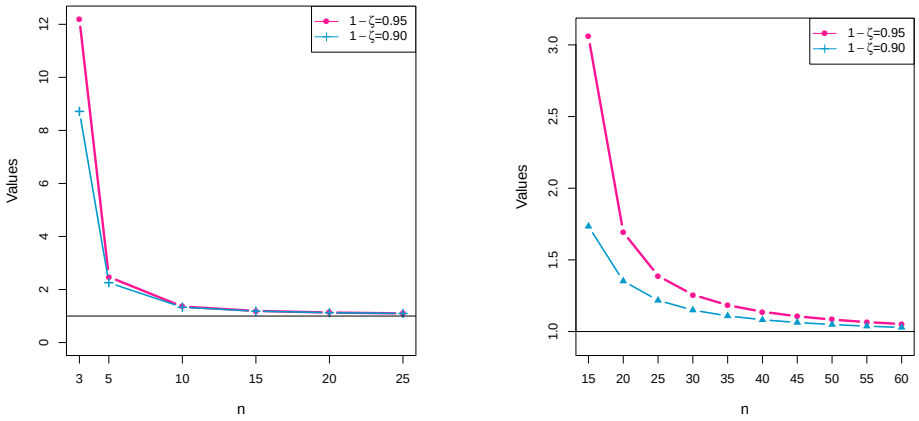
$$\text{Area}(\mathscr{R}^{\text{Approx}}) = \iint\limits_{\mathscr{R}^{\text{Approx}}} \mathrm{d}\mu\,\mathrm{d}\nu = \Big[\int_c^d \sqrt{\frac{4\chi^2_\zeta(2)\nu^2-2n(1-\nu)^2}{n\nu}}\mathrm{d}\nu\Big]\frac{s^3}{n\sqrt{n}},$$

where, $c = \dfrac{n - \sqrt{2n\chi^2_\zeta(2)}}{n - 2\chi^2_\zeta(2)}$ and $d = \dfrac{n + \sqrt{2n\chi^2_\zeta(2)}}{n - 2\chi^2_\zeta(2)}$ for $n > 2\chi^2_\zeta(2)$, and the integral is taken over v, where the term inside the radical is positive.

The areas of the regions $\mathscr{R}^{\text{Mood}}$, $\mathscr{R}^{\text{Approx}}$, and $\mathscr{R}^{\text{Bayes}}$ (corresponding to the non-informative prior with $k = d = 0$ and $n_0 = 0$) are independent of the sample mean $\bar{y}$ and can be expressed as constant multiples of s^3. Consequently, the relative areas, Area($\mathscr{R}^{\text{Mood}}$)/Area($\mathscr{R}^{\text{Bayes}}$) and Area($\mathscr{R}^{\text{Approx}}$)/Area($\mathscr{R}^{\text{Bayes}}$), are non-random and independent of s^3, allowing them to be compared directly. These relative areas depend only on the sample size n and the confidence level $1-\zeta$.

Figure 1(a) shows the relative area Area($\mathscr{R}^{\text{Mood}}$)/Area($\mathscr{R}^{\text{Bayes}}$) for various sample sizes n and two confidence levels, 90% and 95%, for $1-\zeta$. Figure 1(b) displays the relative area Area($\mathscr{R}^{\text{Approx}}$)/Area($\mathscr{R}^{\text{Bayes}}$) for moderate and large sample sizes n, with the same confidence levels, 90% and 95%, for $1-\zeta$.

From Figure 1, it can be observed that both relative areas exceed 1, indicating that the Bayesian region $\mathscr{R}^{\text{Bayes}}$ provides a smaller area than the regions $\mathscr{R}^{\text{Mood}}$ and $\mathscr{R}^{\text{Approx}}$. For large sample sizes, the relative areas approach 1, indicating that the areas become close to each other.



(a) Area($\mathscr{R}^{\text{Mood}}$)/Area($\mathscr{R}^{\text{Bayes}}$) (b) Area($\mathscr{R}^{\text{Approx}}$)/Area($\mathscr{R}^{\text{Bayes}}$)

Figure 1: Relative areas of the regions.

5 Data analysis

This section presents a real data example for illustration and comparison. The following data set, taken from (Montgomery et al., 2010, p. 29), shows the compressive strengths in pounds per square inch (PSI) of 80 specimens of a new aluminum-lithium alloy undergoing evaluation as a possible material for aircraft structural elements. The data in Table 1 have been divided by 10 (10 PSI).

The fit of the normal distribution to these data can be checked using the Kolmogorov-Smirnov (KS) test. The K-S statistic and respective p-value (based on the MLEs $\widehat{\mu} = 16.266$

Table 1: Compressive strength of 80 aluminum-lithium alloy specimens (divided by 10).

10.5	22.1	18.3	18.6	12.1	18.1	18.0	14.3	9.7	15.4	15.3	17.4	12.0	16.8	16.7	14.1
24.5	22.8	17.4	19.9	18.1	15.8	17.6	11.0	16.3	13.1	15.4	11.5	16.0	20.8	15.8	13.3
20.7	18.0	19.0	19.3	19.4	13.3	15.6	12.3	13.4	17.8	7.6	16.7	18.4	13.5	22.9	14.6
21.8	15.7	10.1	17.1	16.5	17.2	15.8	16.9	19.9	15.1	14.2	16.3	14.5	17.1	14.8	15.8
16.0	17.5	14.9	8.7	16.0	23.7	15.0	13.5	19.6	20.1	20.0	17.6	15.0	17.0	11.8	14.9

and $\widehat{\nu} = 11.263$) are KS$= 0.057$ and p-value$= 0.956$, which indicate that the normal distribution provides a very good fit to these data. A normal probability plot can also be used to assess how well the normal distribution fits the data. Figure 2 shows the normal Q-Q plot for the compressive strength data. Since most of the points lie close to the reference line, we conclude that the normal distribution is an appropriate model for this data set.

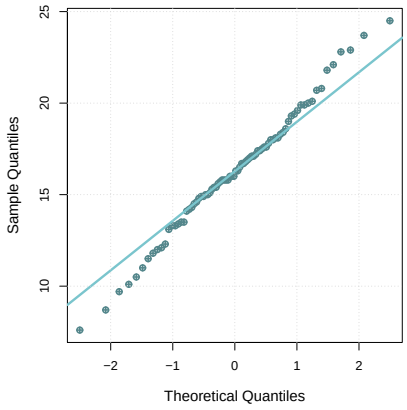


Figure 2: Normal Q-Q plot for the compressive strength data.

We obtain the Bayesian regions for (μ, ν) based on the above data. For comparison, the Mood exact confidence region and the approximate confidence region are also obtained. To obtain Bayesian regions, we use the joint prior distribution given in (2) with $k = d = 0$ and $n_0 = 0$. In other words, we use a non-informative prior assuming that no information about the parameters is available. Figure 3 shows the graph of the posterior density $h(\mu, \nu \mid \mathbf{y})$ based on noninformative prior for the compressive strength data, which is unimodal.

For the compressive strength data and using Section 2, the 95% Mood exact region and 95% approximate region for (μ, ν) are as follows:

$$\mathscr{R}^{\text{Mood}} = \left\{(\nu,\mu):\ 8.211 < \nu < 16.815,\ 16.266 - 2.236\sqrt{\nu} < \mu < 16.266 + 2.236\sqrt{\nu}\right\},$$
$$\mathscr{R}^{\text{Approx}} = \left\{(\mu,\nu):\ 7.102\,(16.266-\mu)^2 + 0.315\,(0.630-\nu)^2 \le 5.991\right\}.$$

The regions $\mathscr{R}^{\text{Mood}}$ and $\mathscr{R}^{\text{Approx}}$ have areas of 15.143 and 15.017, respectively.

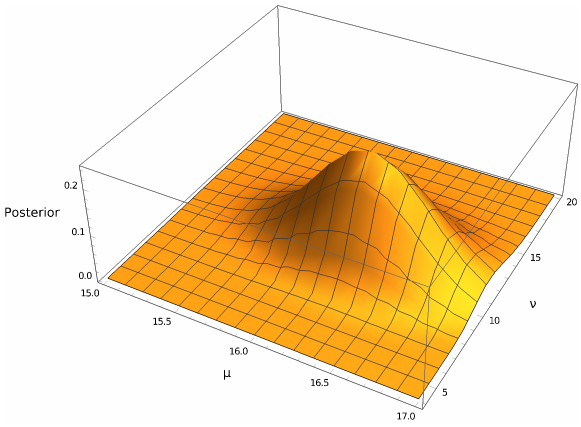


Figure 3: Posterior density of (μ,ν) given $\mathbf{y}$.

Using the method presented in Section 3.2, the 95% balanced Bayesian region for (ν,μ) is

$$\mathscr{R}^{\text{Bayes}} = \Big\{(\nu,\mu):\ 8.124 < \nu < 16.56,\ 16.266 - 0.250\sqrt{\nu} < \mu < 16.266 + 0.250\sqrt{\nu}\Big\},$$

with an area of 14.748.

The 95% HPD region for (μ,ν) is

$$\mathscr{R}^{\text{HPD}} = \big\{(\mu,\nu):\ h(\mu,\nu \mid \mathbf{y}) \geq 0.01061\big\},$$

where $q_{0.95} = 0.01061$ is the 0.05-quantile of the random variable $h(\mu,\nu \mid \mathbf{y})$ obtained using Algorithm 1 for large $\mathscr{M} = 500000$. The HPD region's area is 13.194. Therefore, the amount of reduction in the area of $\mathscr{R}^{\text{HPD}}$ relative to $\mathscr{R}^{\text{Mood}}$ is 12.86%, i.e., $1 - \frac{\text{Area}(\mathscr{R}^{\text{HPD}})}{\text{Area}(\mathscr{R}^{\text{Mood}})} = 12.86\%$ and relative to $\mathscr{R}^{\text{Approx}}$ is 12.14% and relative to $\mathscr{R}^{\text{Bayes}}$ is 10.53%. As expected, the amount of area reduction for $\mathscr{R}^{\text{HPD}}$ is significant.

For better comparison, the 95% regions for (μ,ν) are illustrated in Figures 4 and 5. Figure 4 shows these regions separately, while Figure 5 shows the boundaries of these regions (in mixed form). Clearly, the Bayesian regions are smaller than the classical regions (Mood's exact region and approximate region), and the HPD region provides the smallest area.

Now, for comparison, let us consider an "informative prior" case in (2) with $k = 15, d = 5, m = 16$ and $n_0 = 2$. In this case, the 95% Bayesian region for (ν,μ) is

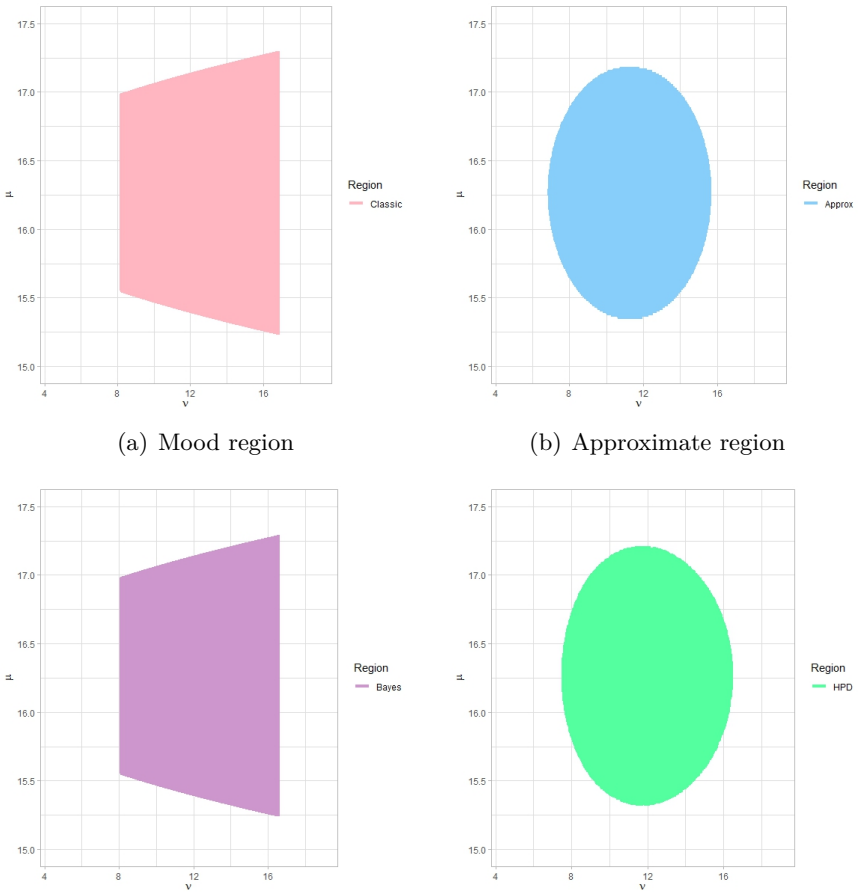
$$\mathscr{R}_*^{\text{Bayes}} = \Big\{(\nu,\mu):\ 7.054 < \nu < 13.552,\ 16.259 - 0.027\sqrt{\nu} < \mu < 16.259 + 0.027\sqrt{\nu}\Big\},$$

with an area of 10.258.

Also, the 95% HPD Bayesian region for (μ,ν) is

$$\mathscr{R}_*^{\text{HPD}} = \big\{(\mu,\nu):\ h(\mu,\nu \mid \mathbf{y}) \geq 0.01530\big\}.$$

The HPD region's area is 9.245. Figure 6 presents the 95% balanced Bayesian and HPD regions for both informative and non-informative prior cases using the compressive strength data. The figure shows that the "informative prior" cases result in smaller areas compared to the "non-informative prior" cases.



(a) Mood region (b) Approximate region

(c) Balanced Bayesian region (d) HPD region

Figure 4: The 95% different regions for (μ, ν).

6 Conclusion

In this paper, we proposed Bayesian credible sets for the mean and variance of a normal distribution. Using the conditional posterior distributions of the parameters, we derived the balanced Bayesian region. A three-step algorithm was used to generate posterior samples and obtain the HPD region. The results showed that the proposed Bayesian regions have smaller areas than the classical exact and approximate confidence regions, and therefore provide a more precise joint description of the uncertainty associated with the mean and variance.

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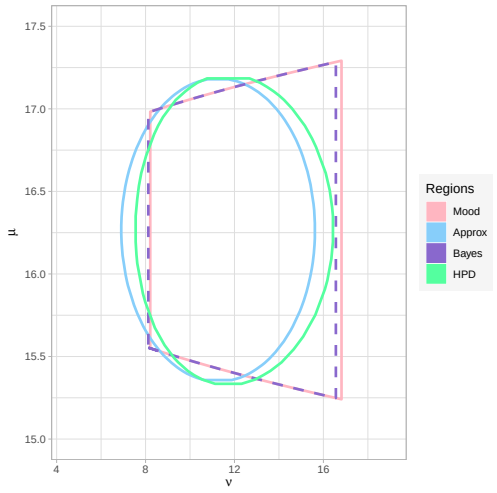
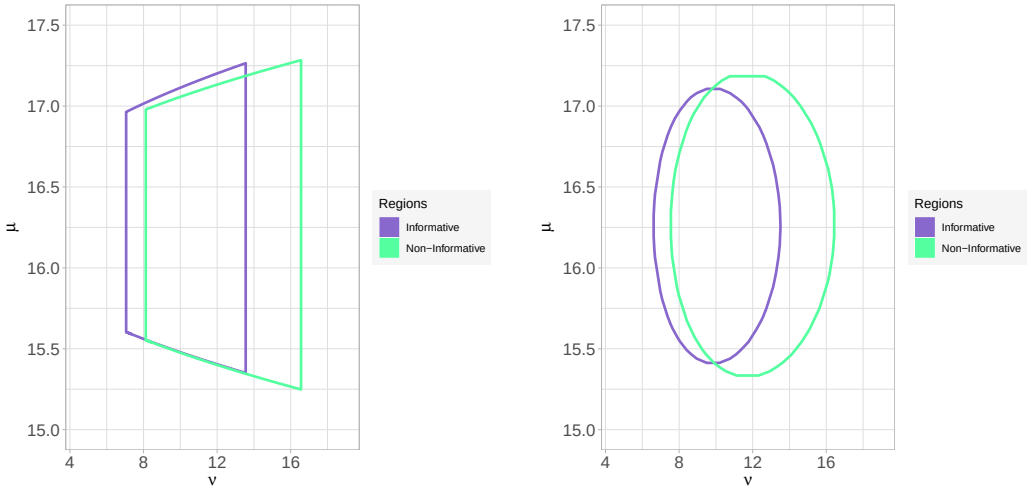


Figure 5: Boundaries of all four 95% regions for (μ, ν).



(a) Balanced Bayesian regions

(b) HPD regions

Figure 6: The 95% Bayesian regions based on informative and non-informative priors.

Data availability statement

All data used in this study are provided in the paper and are available in the published literature.

Disclosure statement

The authors declare no conflicts of interest.

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